

# EXCEPTIONAL COLLECTIONS ON ISOTROPIC GRASSMANNIANS

ALEXANDER KUZNETSOV AND ALEXANDER POLISHCHUK

ABSTRACT. We introduce a new construction of exceptional objects in the derived category of coherent sheaves on a compact homogeneous space of a semisimple algebraic group and show that it produces exceptional collections of the length equal to the rank of the Grothendieck group on homogeneous spaces of all classical groups.

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## 1. INTRODUCTION

The study of derived categories of coherent sheaves on algebraic varieties has been an increasingly popular subject in algebraic geometry. One of important devices relevant for this study is the notion of an exceptional collection (see 1.1 below). In the present paper we give a new general construction of such collections in the derived categories of compact homogeneous spaces of semisimple algebraic groups and show that for classical groups it gives exceptional collections of maximal length.

**1.1. An overview of exceptional collections on homogeneous varieties.** Let  $k$  be a base field which we assume to be algebraically closed of characteristic 0. Recall that an object  $E$  of a  $k$ -linear triangulated category  $\mathcal{T}$  is exceptional, if

$$\mathrm{Ext}^\bullet(E, E) = k$$

(that is  $E$  is simple and has no higher self-Ext's). An ordered collection  $E_1, \dots, E_m$  in  $\mathcal{T}$  is an exceptional collection, if each  $E_i$  is exceptional and

$$\mathrm{Ext}^\bullet(E_i, E_j) = 0$$

for all  $i > j$ . Finally, an exceptional collection  $E_1, \dots, E_m$  is full, if the smallest triangulated subcategory of  $\mathcal{T}$  containing all the objects  $E_1, \dots, E_m$  is  $\mathcal{T}$  itself.

The simplest geometrical example of a full exceptional collection is the collection

$$\{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1), \mathcal{O}(n)\} \in \mathcal{D}(\mathbb{P}^n)$$

in the bounded derived category of coherent sheaves on  $\mathbb{P}^n$  constructed by Beilinson in his pioneering work [Bei]. After that a vast number of exceptional collections was constructed by Kapranov in [Kap]. In fact, he constructed full exceptional collections of vector bundles on all homogeneous spaces of the simple algebraic groups of type  $A$  and on quadrics (which are special homogeneous spaces of types  $B$  and  $D$ ). This naturally led to the following conjecture.

**Conjecture 1.1.** *If  $\mathbf{G}$  is a semisimple algebraic group and  $\mathbf{P} \subset \mathbf{G}$  is a parabolic subgroup of  $\mathbf{G}$  then there is a full exceptional collection of vector bundles in  $\mathcal{D}(\mathbf{G}/\mathbf{P})$ .*

Up to now only partial results in this direction were obtained. Below we list all minimal homogeneous varieties of simple groups (corresponding to maximal parabolic subgroups) for which a full exceptional collection was constructed. Recall that simple algebraic groups are classified by Dynkin diagrams that fall into types  $A, B, C, D, E, F$  and  $G$ . Maximal parabolic subgroups correspond to vertices of Dynkin diagrams for which we use the standard numbering (see [Bou]). Thus, we denote by  $\mathbf{P}_i$  the maximal parabolic subgroup corresponding to the vertex  $i$ .

**Type  $A_n$ :** A full collection was constructed by Kapranov in [Kap].

**Type  $B_n$ :** For  $\mathbf{P} = \mathbf{P}_1$  (so that  $\mathbf{G}/\mathbf{P} = Q^{2n-1}$ , a quadric of dimension  $2n-1$ ) a full exceptional collection was constructed by Kapranov in [Kap]. For  $\mathbf{P} = \mathbf{P}_2$  (so that  $\mathbf{G}/\mathbf{P} = \text{OGr}(2, 2n+1)$ , the Grassmannian of lines on  $Q^{2n-1}$ ) a full exceptional collection was constructed in [K08]. For  $n = 4$  and  $\mathbf{P} = \mathbf{P}_4$  (so that  $\mathbf{G}/\mathbf{P} = \text{OGr}(4, 9) = \text{OGr}_+(5, 10)$ ) a full exceptional collection was constructed in [K06].

**Type  $C_n$ :** For  $\mathbf{P} = \mathbf{P}_1$  (so that  $\mathbf{G}/\mathbf{P} = \mathbb{P}^{2n-1}$ ), the Beilinson's collection works. For  $\mathbf{P} = \mathbf{P}_2$  (so that  $\mathbf{G}/\mathbf{P} = \text{SGr}(2, 2n)$ , the Grassmannian of isotropic lines) a full exceptional collection was constructed in [K08]. For  $n = 3, 4, 5$  and  $\mathbf{P} = \mathbf{P}_n$  (so that  $\mathbf{G}/\mathbf{P} = \text{SGr}(n, 2n)$ , the Lagrangian Grassmannian) full exceptional collections were constructed in [S01] and [PS].

**Type  $D_n$ :** For  $\mathbf{P} = \mathbf{P}_1$  (so that  $\mathbf{G}/\mathbf{P} = Q^{2n-2}$ , a quadric of dimension  $2n-2$ ) a full exceptional collection was constructed by Kapranov in [Kap]. For  $\mathbf{P} = \mathbf{P}_2$  (so that  $\mathbf{G}/\mathbf{P} = \text{OGr}(2, 2n)$ , the Grassmannian of isotropic lines on  $Q^{2n-2}$ ) an almost full exceptional collection was constructed in [K08].

**Type  $E_n$ :** For  $n = 6$  and  $\mathbf{P} = \mathbf{P}_1$  (or  $\mathbf{P} = \mathbf{P}_6$ ) an exceptional collection was constructed by Manivel in [Man]. However its fullness is not proved.

**Type  $F_4$ :** For  $\mathbf{P} = \mathbf{P}_4$  (so that  $\mathbf{G}/\mathbf{P}$  is a hyperplane section of  $E_6/\mathbf{P}_1$ ) an exceptional collection can be constructed by restricting the Manivel's collection.

**Type  $G_2$ :** For  $\mathbf{P} = \mathbf{P}_1$  (so that  $\mathbf{G}/\mathbf{P} = Q^5$ ) the Kapranov's collection works. For  $\mathbf{P} = \mathbf{P}_2$  a full exceptional collection was constructed in [K06].

**1.2. The statement of results.** The main result of the present paper can be formulated as follows. Let us say that an exceptional collection on an algebraic variety is of **expected length**, if the length equals the rank of the Grothendieck group  $\text{rk}(K_0(X))$ . Note that if  $K_0(X)$  is a free abelian group then this implies that the corresponding classes generate  $K_0(X)$ .

Let us say that a simple group  $\mathbf{G}$  is of type  $BCD$  if its type is either  $B_n$ , or  $C_n$ , or  $D_n$ .

**Theorem 1.2.** *Let  $\mathbf{G}$  be a simply connected simple group of type  $BCD$ . Then for each maximal parabolic subgroup  $\mathbf{P} \subset \mathbf{G}$  there is an exceptional collection of expected length in  $\mathcal{D}(\mathbf{G}/\mathbf{P})$  the objects of which have a  $\mathbf{G}$ -equivariant structure.*

Note that the existence of a  $\mathbf{G}$ -equivariant structure here is a general result (see [P11, Lem. 2.2]) but also comes naturally from the construction. The  $\mathbf{G}$ -equivariant structure on objects of our collections allows to construct a relative exceptional collection on any fibration with fiber  $\mathbf{G}/\mathbf{P}$  (see [S07, Thm. 3.1]).

**Corollary 1.3.** *Let  $\mathbf{G}$  and  $\mathbf{P}$  be as in Theorem 1.2, and let  $\mathcal{G} \rightarrow X$  be a principal  $\mathbf{G}$ -bundle, where  $X$  is an algebraic variety. Consider the corresponding fibration  $Y = \mathcal{G} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) \rightarrow X$ . Then there is a semiorthogonal decomposition of  $\mathcal{D}^b(Y)$  consisting of  $\text{rk}(K_0(\mathbf{G}/\mathbf{P}))$  subcategories, each equivalent to  $\mathcal{D}^b(X)$ , and possibly an additional subcategory. In particular, if  $X$  has an exceptional collection of the expected length then so does  $Y$ .*

Now note that for an arbitrary (not maximal) parabolic subgroup  $\mathbf{P} \subset \mathbf{G}$  the homogeneous space  $\mathbf{G}/\mathbf{P}$  has a structure of an iterated fibration with fibers of the form  $\mathbf{G}_i/\mathbf{P}_i$ , where  $\mathbf{G}_i$  are semisimple algebraic

groups and  $\mathbf{P}_i \subset \mathbf{G}_i$  are maximal parabolic subgroups. Moreover, if  $\mathbf{G}$  is of type  $BCD$  then all  $\mathbf{G}_i$  are of type  $BCD$  as well. So, applying Corollary 1.3 several times we conclude that

**Corollary 1.4.** *If  $\mathbf{G}$  is a simple group of type  $BCD$  and  $\mathbf{P} \subset \mathbf{G}$  is a (not necessary maximal) parabolic subgroup then there is an exceptional collection of expected length in  $\mathcal{D}(\mathbf{G}/\mathbf{P})$ .*

We conjecture that the exceptional collections we construct are full and possess further nice properties that we checked in some special cases (see Conjecture 1.9).

Finally, we would like to stress that our construction of an exceptional collection is quite general: we use special properties of types  $BCD$  only in some computations. So, we hope that the approach of this paper can be used to construct full exceptional collections for all the remaining homogeneous spaces (i.e., for the exceptional groups  $E_6$ ,  $E_7$ ,  $E_8$  and  $F_4$ ).

**1.3. An overview of the construction.** The main part of any construction of an exceptional collection is to find sufficiently many exceptional objects. For a homogeneous variety it is natural to try equivariant bundles.

Note that when we fix a type of a simple group we have several choices of the group itself, ranging from simply connected to adjoint cases. The simply connected group has the most rich category of equivariant bundles. On the other hand, the homogeneous varieties for all groups of a given type are the same. Because of this *from now on we will assume that  $\mathbf{G}$  is simply connected*.

Recall that there is an equivalence of the category of  $\mathbf{G}$ -equivariant coherent sheaves on  $\mathbf{G}/\mathbf{P}$  and the category of representations of  $\mathbf{P}$ :

$$\mathrm{Coh}^{\mathbf{G}}(\mathbf{G}/\mathbf{P}) \cong \mathrm{Rep} \mathbf{P},$$

see [BK]. In fact, it is an equivalence of tensor abelian categories. In particular, each representation of  $\mathbf{P}$  can be considered as a vector bundle on  $X = \mathbf{G}/\mathbf{P}$ . The group  $\mathbf{P}$  is not reductive, so its representation theory is rather complicated. Let us start by considering the semisimple part of the category,  $\mathrm{Rep}^{\mathrm{ss}} \mathbf{P}$ . This semisimple part is the subcategory of representations on which the unipotent radical  $\mathbf{U}$  acts trivially. Thus, it is equivalent to  $\mathrm{Rep} \mathbf{L}$ , where

$$\mathbf{L} = \mathbf{P}/\mathbf{U}$$

is the Levi quotient (the equivalence  $\mathrm{Rep}^{\mathrm{ss}} \mathbf{P} \cong \mathrm{Rep} \mathbf{L}$  is given by the restriction functor for the projection  $\mathbf{P} \rightarrow \mathbf{L}$ ). The Levi group  $\mathbf{L}$  is reductive, and its weight lattice  $P_{\mathbf{L}}$  is canonically isomorphic to the weight lattice  $P_{\mathbf{G}}$  of the group  $\mathbf{G}$ . We denote the cone of  $\mathbf{L}$ -dominant weights by  $P_{\mathbf{L}}^+ \subset P_{\mathbf{L}}$  and the cone of  $\mathbf{G}$ -dominant weights by  $P_{\mathbf{G}}^+ \subset P_{\mathbf{G}}$ . Irreducible representations of  $\mathbf{L}$  are parameterized by their highest weights which are  $\mathbf{L}$ -dominant. For each  $\mathbf{L}$ -dominant weight  $\lambda \in P_{\mathbf{L}}^+$  we denote by  $V_{\mathbf{L}}^{\lambda}$  the corresponding irreducible representation of  $\mathbf{L}$ , as well as its restriction to  $\mathbf{P}$ , and by  $\mathcal{U}^{\lambda}$  the corresponding  $\mathbf{G}$ -equivariant bundle on  $X = \mathbf{G}/\mathbf{P}$ .

In type  $A$  there are sufficiently many exceptional bundles among the  $\mathcal{U}^{\lambda}$ 's, so one can construct an exceptional collection of expected length out of them. However, for other types the situation is not so nice. Although all  $\mathcal{U}^{\lambda}$  are exceptional in the equivariant derived category  $\mathcal{D}^{\mathbf{G}}(X)$ , it turns out that only few of them are exceptional in  $\mathcal{D}(X)$ . For example, in the case when  $\mathbf{G}$  is of type  $C_n$  and  $\mathbf{P} = \mathbf{P}_n$ , so that  $X = \mathrm{SGr}(n, 2n)$  (the Lagrangian Grassmannian), one can check that  $\mathcal{U}^{\lambda}$  is exceptional if and only if

$$\lambda = \omega_i + t\omega_n,$$

where  $\omega_i$  is the fundamental weight of the vertex  $i$  of the Dynkin diagram. Since also  $K_X = \mathcal{U}^{-(n+1)\omega_n}$ , one can deduce easily that the maximal possible length of an exceptional collection for  $X$  consisting of vector bundles of the form  $\mathcal{U}^{\lambda}$  is  $n(n+1)$  (we have  $n$  choices for  $i$  and  $n+1$  choices for  $t$  in the above formula for  $\lambda$ ), whereas  $\mathrm{rk}(K_0(X)) = 2^n$ . So, for  $n \geq 5$  we have no chance to find an exceptional

collection of expected length consisting only of  $\mathcal{U}^\lambda$ . In other words, we need to introduce another class of  $\mathbf{P}$ -modules. In fact, this is the most interesting problem discussed in this paper.

To explain how we do it let us return to the example of the group  $\mathbf{G}$  of type  $C_n$  and of  $\mathbf{P} = \mathbf{P}_n$ . Recall that in this case the lattice of weights is

$$P_{\mathbf{L}} = P_{\mathbf{G}} = \mathbb{Z}^n = \{(\lambda_1, \dots, \lambda_n)\},$$

and the dominant cones can be described as

$$P_{\mathbf{G}}^+ = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0\}, \quad P_{\mathbf{L}}^+ = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$$

(the Levi group  $\mathbf{L}$  in this case is isomorphic to  $\mathrm{GL}_n$ ). Take any integer  $0 \leq a \leq n$  and consider a subset (a block)

$$B_a = \{n \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_a \geq \lambda_{a+1} = \dots = \lambda_n = a\}.$$

Its elements can be viewed as Young diagrams inscribed in  $(n-a) \times a$  rectangle. In particular,

$$\#B_a = \binom{n}{a}.$$

It turns out that for the weights  $\lambda, \mu$  within such a block  $B = B_a$  the following amusing property is satisfied: the canonical map

$$\bigoplus_{\nu \in B} \mathrm{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathrm{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \rightarrow \mathrm{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) \quad (\star)$$

is an isomorphism (here  $\mathrm{Ext}_{\mathbf{G}}$  stands for the  $\mathrm{Ext}$  groups in the derived category  $\mathcal{D}^{\mathbf{G}}(X)$  of  $\mathbf{G}$ -equivariant coherent sheaves on  $X$ , and the map is given by the composition of equivariant  $\mathrm{Ext}$ 's with  $\mathrm{Hom}$ 's).

Now, having this property one can formally check that

- considering  $\{\mathcal{U}^\lambda\}_{\lambda \in B_a}$  as a (nonfull) exceptional collection in  $\mathcal{D}^{\mathbf{G}}(X)$ ,
- passing to the *right dual exceptional collection*  $\{\mathcal{E}^\lambda\}_{\lambda \in B_a}$  in  $\mathcal{D}^{\mathbf{G}}(X)$ , and then
- forgetting the equivariant structure on all  $\mathcal{E}^\lambda$

one obtains an exceptional collection,  $\{\mathcal{E}^\lambda\}_{\lambda \in B_a}$  in  $\mathcal{D}(X)$  which generates the same subcategory as the original (non-exceptional) collection  $\{\mathcal{U}^\lambda\}$ . This strange procedure can be considered as the central construction of the paper. To make it work in general we introduce a notion of an **exceptional block**  $B \subset P_{\mathbf{L}}^+$ . By definition, an exceptional block is a subset  $B \subset P_{\mathbf{L}}^+$  of  $\mathbf{L}$ -dominant weights such that the morphism  $(\star)$  is an isomorphism. The procedure described above produces an exceptional collection  $\{\mathcal{E}^\lambda\}_{\lambda \in B}$  generating the category

$$\mathcal{A}_B := \langle \mathcal{U}^\lambda \rangle_{\lambda \in B}.$$

However, in general one cannot find an exceptional block of expected length, so to obtain an exceptional collection of expected length we combine several exceptional blocks which are semiorthogonal, i.e. all  $\mathrm{Ext}$ 's between blocks in the order-decreasing direction vanish. For example, for  $\mathbf{G}$  of type  $C_n$  and  $\mathbf{P} = \mathbf{P}_n$  we take all the blocks  $B_a$  described above. Note that the total number of exceptional objects then is  $\sum_{a=0}^n \binom{n}{a} = 2^n$ .

Our construction depends on several choices (subject to one restriction) that we are going to explain now. Let  $D = D_{\mathbf{G}}$  be the Dynkin diagram of  $\mathbf{G}$ . Denote by  $\beta$  the simple root (a vertex of  $D$ ) corresponding to the maximal parabolic  $\mathbf{P}$ , and by  $\xi$  the corresponding fundamental weight of  $\mathbf{G}$ .

- (C1) We choose a connected component of  $D \setminus \beta$ , called the **outer component** and denoted by  $D_{\mathrm{out}}$ . We also allow  $D_{\mathrm{out}}$  to be empty.

The restriction is

- (R) If  $D_{\mathrm{out}}$  is nonempty then it is a Dynkin diagram of type  $A$ .

We denote the complement of  $\beta$  and  $D_{\text{out}}$  by  $D_{\text{inn}}$

$$D_{\text{inn}} = D_{\mathbf{G}} \setminus (D_{\text{out}} \cup \beta)$$

and call it the inner component of  $D_{\mathbf{G}}$ . We consider the simply connected subgroups

$$\mathbf{L}_{\text{out}}, \mathbf{L}_{\text{inn}} \subset \mathbf{L}$$

corresponding to the subdiagrams  $D_{\text{out}}, D_{\text{inn}} \subset D \setminus \beta = D_{\mathbf{L}}$  and denote by

$$i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{L}, \quad o : \mathbf{L}_{\text{out}} \rightarrow \mathbf{L}$$

the embeddings. Abusing the notation we denote the embeddings of these subgroups into  $\mathbf{G}$  by the same letters. Our restriction on  $D_{\text{out}}$  means that  $\mathbf{L}_{\text{out}} \simeq \text{SL}_k$  for some  $k \geq 1$ .

The next choice is the following.

(C2) We choose a chain of vertices  $1, 2, \dots, b \in D_{\text{out}}$  starting at a terminal vertex 1 of the diagram  $D$  and ending at the vertex  $b$  which is adjacent to  $\beta$ .

Here in fact we have at most two different choices (since we assumed that  $D_{\text{out}}$  is of type  $A$ ). Note that for classical groups (types  $B$ ,  $C$  and  $D$ ) the standard numbering of vertices has the required property.

We have the following decreasing chain of Dynkin subdiagrams in  $D_{\mathbf{G}}$ :

$$D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}$$

for  $a = 0, 1, \dots, b$ . Let

$$h_a : \mathbf{H}_a \rightarrow \mathbf{G}$$

be the embedding of the simply connected subgroup corresponding to the subdiagram  $D_a$ . Note that  $\mathbf{L}_{\text{inn}} \subset \mathbf{H}_a$  since  $D_{\text{inn}} \subset D_a$ , so the embedding  $i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{G}$  factors through an embedding  $\mathbf{L}_{\text{inn}} \rightarrow \mathbf{H}_a$  that we will also denote by  $i$ . If  $\mathbf{K}$  is any of the groups  $\mathbf{G}, \mathbf{L}, \mathbf{L}_{\text{inn}}, \mathbf{L}_{\text{out}}, \mathbf{H}_a$  then we denote by  $P_{\mathbf{K}}$  (resp.,  $\mathbf{W}_{\mathbf{K}}$ ) the corresponding weight lattice (resp., Weyl group).

The third choice is the following.

(C3) For each  $a = 0, 1, \dots, b$  we choose a strictly dominant weight  $\delta_a \in P_{\mathbf{H}_a}^+$ .

For each  $a = 0, 1, \dots, b$  we define a polyhedron in  $P_{\mathbf{H}_a} \otimes \mathbb{R}$  by

$$\mathbf{R}_{\delta_a} = \{\lambda \in P_{\mathbf{H}_a} \otimes \mathbb{R} \mid \forall w \in \mathbf{W}_{\mathbf{H}_a} \quad (\lambda, w\delta_a) \leq (\rho_{\mathbf{H}_a}, \delta_a)\}$$

We will refer to  $\mathbf{R}_{\delta_a}$  as the core in  $P_{\mathbf{H}_a} \otimes \mathbb{R}$ .

We denote by  $r_a$  the index of the Grassmannian  $\mathbf{H}_a/(\mathbf{P} \cap \mathbf{H}_a)$  (the maximal integer dividing the canonical class of  $\mathbf{H}_a/(\mathbf{P} \cap \mathbf{H}_a)$  in the Picard group). One can easily see that this sequence is strictly decreasing

$$r = r_0 > r_1 > \dots > r_{b-1} > r_b > 0.$$

Let us denote by  $\theta$  an element of  $P_{\mathbf{L}} \otimes \mathbb{Q}$  such that

$$\theta \in \langle \omega_1, \dots, \omega_{k-1} \rangle^\perp \cap \text{Ker } i^*, \quad \text{and} \quad (\theta, \xi) = 1$$

(it is easy to see that such  $\theta$  always exists and is unique). Note that the set  $(\theta, P_{\mathbf{L}})$  of all scalar products of  $\theta$  with weights of  $\mathbf{L}$  is a sublattice of  $\mathbb{Q}$  containing  $\mathbb{Z}$ . For example, let us describe  $(\theta, P_{\mathbf{L}})$  for all classical groups (using the standard numbering of vertices). If  $\mathbf{G}$  is of type  $C$  then  $\theta = \omega_k - \omega_{k-1}$  and  $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$ ; if  $\mathbf{G}$  is of type  $B$  and  $k \leq n-1$  or  $\mathbf{G}$  is of type  $D$  and  $k \leq n-2$  then  $\theta = \omega_k - \omega_{k-1}$  and  $(\theta, P_{\mathbf{L}}) = \frac{1}{2}\mathbb{Z}$ . Finally, if  $\mathbf{G}$  is of type  $B$  and  $k = n$  or  $\mathbf{G}$  is of type  $D$  and  $k \geq n-1$  then  $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$ . On the other hand, if  $\mathbf{G}$  is of type  $E_6$  then the lattice  $(\theta, P_{\mathbf{L}})$  can be equal to  $\frac{1}{4}\mathbb{Z}$ ,  $\frac{1}{5}\mathbb{Z}$  or  $\frac{1}{6}\mathbb{Z}$  depending on the root  $\beta$ .

Consider the following segment of the lattice  $(\theta, P_{\mathbf{L}})$ :

$$J = \{j \in (\theta, P_{\mathbf{L}}) \mid 0 \leq j < r\}.$$

For each  $j \in J$  we will construct an exceptional block  $\bar{B}_j \subset P_{\mathbf{L}}^+$ , such that the union of the corresponding exceptional collections over all  $j \in J$  will be an exceptional collection of expected length in  $\mathcal{D}(X)$ .

For each  $j \in J$  we define an integer  $a(j)$ ,  $0 \leq a(j) \leq b$  by the condition

$$r - r_{a(j)} \leq j < r - r_{a(j)+1},$$

For brevity we will write  $\mathbf{H}_j = \mathbf{H}_{a(j)}$ ,  $h_j = h_{a(j)}$  and  $\mathbf{R}_j = \mathbf{R}_{\delta_{a(j)}}$ .

First, we construct for each  $j \in J$  a block  $B_j$  of the form

$$B_j = B_j^{\text{out}} + j\xi + i_*(B_j^{\text{inn}})$$

with  $B_j^{\text{out}} \in \text{Ker } h_j^* = \langle \omega_1, \omega_2, \dots, \omega_{a(j)-1} \rangle$  (called the outer part of the block), and  $B_j^{\text{inn}} \subset P_{\mathbf{L}_{\text{inn}}}$  (called the inner part of the block). The inner part is given by

$$B_j^{\text{inn}} = \left\{ \nu \in P_{\mathbf{L}_{\text{inn}}}^+ \mid \begin{array}{l} (1) \quad \rho_{\mathbf{H}_j} \pm 2i_*(w\nu) \in \mathbf{R}_j \text{ for all } w \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \\ (2) \quad j\xi + i_*\nu \in P_{\mathbf{L}} \end{array} \right\}$$

and then the outer part is defined by

$$B_j^{\text{out}} = \left\{ \mu \in \text{Ker } h_j^* \cap P_{\mathbf{G}}^+ \mid \begin{array}{l} \rho_{\mathbf{H}_j} - h_j^*(w_{\mathbf{L}_{\text{out}}}\mu) - i_*(w_{\mathbf{L}_{\text{inn}}}\nu) + i_*(w'_{\mathbf{L}_{\text{inn}}}\nu') \in \mathbf{R}_j \\ \text{for all } \nu, \nu' \in B_j^{\text{inn}}, w_{\mathbf{L}_{\text{out}}} \in \mathbf{W}_{\mathbf{L}_{\text{out}}}, \text{ and } w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \end{array} \right\}.$$

Note that by definition of  $\theta$  we have  $(\theta, B_j) = j$ . So, the pairing with  $\theta$  gives the ordering of the blocks.

We check that the blocks  $B_j$  constructed above are exceptional provided the group  $\mathbf{G}$  is of type  $BCD$  (for other types one has to modify slightly the definition of the outer part by taking into account some very special elements of the set  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$ , see details in section 5.5). Hence, for each  $j \in J$  the category  $\langle \mathcal{U}^\lambda \rangle_{\lambda \in B_j}$  is generated by an exceptional collection. However, it turns out that these subcategories are not semiorthogonal, so we have to make our blocks slightly smaller. Let  $\mathbf{R}_j^*$  denote the interior of the core  $\mathbf{R}_j$ . We define the subsets  $\bar{B}_j^{\text{inn}} \subset B_j^{\text{inn}}$  for  $j \in J$  recursively (starting from  $j = 0$ ) by

$$\bar{B}_j^{\text{inn}} = \left\{ \nu \in B_j^{\text{inn}} \mid \begin{array}{l} \text{for all } j' < j, \nu' \in \bar{B}_{j'}^{\text{inn}}, \text{ and } w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \\ \text{one has } \rho_{\mathbf{H}_{j'}} - (j - j')\xi - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu' \in \mathbf{R}_{j'}^* \end{array} \right\}.$$

Then we set

$$\bar{B}_j^{\text{out}} = \left\{ \lambda_0 \in B_j^{\text{out}} \mid \begin{array}{l} \text{for all } j' < j, \nu \in \bar{B}_{j'}^{\text{inn}}, \nu' \in \bar{B}_{j'}^{\text{inn}}, w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}}, \text{ and } w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}} \\ \text{one has } \rho_{\mathbf{H}_{j'}} - h_{j'}^*(w_{\mathbf{L}}\lambda_0 + (j - j')\xi) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu' \in \mathbf{R}_{j'}^* \end{array} \right\}$$

and, as before,

$$\bar{B}_j = \bar{B}_j^{\text{out}} + j\xi + i_*\bar{B}_j^{\text{inn}}.$$

Note that  $\bar{B}_j^{\text{out}}$  is a set of linear combinations of fundamental weights  $\omega_1, \dots, \omega_{a(j)}$  with nonnegative coefficients. These can be considered as Young diagrams — a weight  $x_1\omega_1 + \dots + x_a\omega_a$  corresponds to the Young diagram with  $x_i$  rows of length  $i$ . Let us say that the set  $\bar{B}_j^{\text{out}}$  is closed under passing to Young subdiagrams if the corresponding set of Young diagrams is.

Here is a more precise version of our main result. Consider the subcategories

$$\mathcal{A}_j = \langle \mathcal{U}^\lambda \rangle_{\lambda \in \bar{B}_j}.$$

**Theorem 1.5.** (i) For each simply connected simple group  $\mathbf{G}$  the collection of categories

$$\{\mathcal{A}_j\}_{j \in J}$$

constructed above is semiorthogonal.

(ii) For  $j \in J$  such that  $\bar{B}_j^{\text{out}}$  is closed under passing to Young subdiagrams, the block  $\bar{B}_j$  is exceptional.

(iii) If  $\mathbf{G}$  is a group of type  $BCD$  then the choices (C1), (C2), (C3) can be made in such a way that the assumption of (ii) is satisfied for all  $j \in J$  and the resulting exceptional collection

$$(1) \quad \{\mathcal{E}^\lambda\}_{\lambda \in \bar{B}_j, j \in J}$$

in  $\mathcal{D}(X)$  is of expected length.

Note that Theorem 1.2 follows from this.

**Conjecture 1.6.** *The exceptional collections constructed in the Theorem 1.5(iii) are full.*

*Remark 1.7.* We believe that every exceptional collection of expected length on  $\mathbf{G}/\mathbf{P}$  is full. This would follow from a much more general Nonvanishing Conjecture, see [K09, Cor. 9.3].

Recall that an exceptional collection  $E_1, \dots, E_m$  in a triangulated category  $\mathcal{T}$  is **strong**, if

$$\mathrm{Ext}^{\neq 0}(E_i, E_j) = 0$$

for all  $i, j$ . An advantage of a strong exceptional collection is that it gives an equivalence of the category  $\mathcal{T}$  with the derived category of modules over an algebra  $\mathrm{End}(\oplus E_i)$  (for a non-strong collection one has to deal with a DG-algebra). Let us say that an exceptional collection is **pure** if all  $E_i$  are vector bundles.

**Theorem 1.8.** *For the blocks of the collections constructed in Theorem 1.5(i) strongness and purity are equivalent.*

In fact, we conjecture that

**Conjecture 1.9.** *The collections constructed in Theorem 1.5(iii) are pure and blockwise strong.*

We verify this conjecture for all maximal isotropic Grassmannians (symplectic and orthogonal).

**1.4. Further questions.** There are several questions to be investigated.

*Question 1.10.* Is there a way to make choices (C1)–(C3) in a canonical way? Is restriction (R) really necessary for the construction.

It seems that our constructions should work under a certain weakening of the restriction (R) which would allow to construct many nontrivial interesting exceptional collections even for  $\mathcal{D}^b(\mathrm{Gr}(k, n))$ . In particular, it would allow to construct an exceptional collection in  $\mathcal{D}^b(\mathrm{Gr}(k, 2k))$  which is invariant under the outer automorphism. For more details see section 9.6.

*Question 1.11.* Assume that  $\mathbf{G}$  is an exceptional group (types  $E_6, E_7, E_8$  and  $F_4$ ). Is it possible to make the choices (C1)–(C3) in a way analogous to Theorem 1.5(iii), so as to get an exceptional collection of expected length?

Note that in the case of groups of type  $BCD$  this is a result of direct calculation without an a priori explanation. It would be nice to understand the combinatorics behind this coincidence. Recall that the rank of the Grothendieck group  $K_0(\mathbf{G}/\mathbf{P})$  is equal to  $|\mathbf{W}_{\mathbf{G}}/\mathbf{W}_{\mathbf{L}}|$ , so the following question seems natural.

*Question 1.12.* Find a decomposition of the set  $\mathbf{W}_{\mathbf{G}}/\mathbf{W}_{\mathbf{L}} = \bigsqcup_{j \in J} W_j$  and a bijection between the sets  $W_j$  and the sets  $\bar{B}_j$ .

The above decomposition should depend on a chain of subgroups  $\mathbf{H}_b \subset \dots \mathbf{H}_1 \subset \mathbf{H}_0 = \mathbf{G}$ .



**1.5. The structure of the paper.** We start by collecting in section 2 the notation and basic facts about representation theory of algebraic groups.

In section 3 we define exceptional blocks, prove that they produce exceptional collections, investigate their properties, and state a criterion of exceptionality of a block.

In section 4 we discuss strongness and purity of the collections.

In section 5 we define the blocks  $B_j$  and  $\bar{B}_j$  and show that  $(\mathcal{A}_j)$  is a semiorthogonal collection of subcategories.

In section 6 we verify the first part of the criterion — the invariance condition — for  $B_j$  and  $\bar{B}_j$ .

In section 7 we verify the second part of the criterion — the compatibility condition — modulo a technical assumption (that the outer part of each block is closed under passing to Young subdiagrams)

In section 8 we rewrite the definition of the blocks in a more explicit form.

In section 9 we write down the precise choices for classical groups and prove that they give exceptional collections of expected length.

Finally, in the Appendix (section 10) we prove a certain property of representations of the general linear group which is used for the proof of exceptionality of the blocks.

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## 2. PRELIMINARIES

### 2.1. Notation.

#### (1) Groups

- $\mathbf{G}$ , a simple simply connected algebraic group;
- $\mathbf{P} \subset \mathbf{G}$ , a maximal parabolic subgroup;
- $\mathbf{U} \subset \mathbf{P}$ , the unipotent radical;
- $\mathbf{L} = \mathbf{P}/\mathbf{U}$ , the Levi quotient, there is also an embedding  $\mathbf{L} \subset \mathbf{P} \subset \mathbf{G}$ ;
- $\mathbf{L}_{\text{inn}} \subset \mathbf{L}$ , the inner part of  $\mathbf{L}$ , see section 5.2;
- $\mathbf{L}_{\text{out}} \subset \mathbf{L}$ , the outer part of  $\mathbf{L}$ , see section 5.2;
- $\mathbf{H}_a \subset \mathbf{G}$ ,  $\mathbf{L}_{\text{inn}} \subset \mathbf{H}_a$ , a semisimple subgroup, see section 5.2;
- $\mathbf{M}_a = \mathbf{L} \cap \mathbf{H}_a$ , the Levi of  $\mathbf{H}_a$ ;
- $\mathbf{M}_{a,\text{inn}} = \mathbf{L}_{\text{inn}} \cap \mathbf{H}_a = \mathbf{L}_{\text{inn}}$ , the inner part of the Levi of  $\mathbf{H}_a$ ;
- $\mathbf{M}_{a,\text{out}} = \mathbf{L}_{\text{out}} \cap \mathbf{H}_a$ , the outer part of the Levi of  $\mathbf{H}_a$ ;

#### (2) Roots, weights

- $D = D_{\mathbf{G}}$ ,  $D_{\mathbf{L}_{\text{inn}}} = D_{\text{inn}} \subset D$ ,  $D_{\mathbf{L}_{\text{out}}} = D_{\text{out}} \subset D$ ,  $D_{\mathbf{H}_a} = D_a \subset D$ , the Dynkin diagrams;
- $Q_{\mathbf{G}}$ ,  $Q_{\mathbf{L}}$ ,  $Q_{\mathbf{L}_{\text{inn}}}$ ,  $Q_{\mathbf{L}_{\text{out}}}$ ,  $Q_{\mathbf{H}_a}$ , the root lattices;
- $Q_{\mathbf{G}}^+$ ,  $Q_{\mathbf{L}}^+$ ,  $Q_{\mathbf{L}_{\text{inn}}}^+$ ,  $Q_{\mathbf{L}_{\text{out}}}^+$ ,  $Q_{\mathbf{H}_a}^+$ , the cones generated by simple roots;
- $P_{\mathbf{G}}$ ,  $P_{\mathbf{L}} = P_{\mathbf{L}}$ ,  $P_{\mathbf{L}_{\text{inn}}}$ ,  $P_{\mathbf{L}_{\text{out}}}$ ,  $P_{\mathbf{H}_a}$ , the weight lattices;
- $P_{\mathbf{G}}^+ \subset P_{\mathbf{G}}$ ,  $P_{\mathbf{L}}^+ \subset P_{\mathbf{L}}$ ,  $P_{\mathbf{L}_{\text{out}}}^+ \subset P_{\mathbf{L}_{\text{out}}}$ ,  $P_{\mathbf{L}_{\text{inn}}}^+ \subset P_{\mathbf{L}_{\text{inn}}}$ ,  $P_{\mathbf{H}_a}^+ \subset P_{\mathbf{H}_a}$ , the dominant cones;
- $\alpha_i$ , the simple roots;
- $\omega_i$ , the fundamental weights;
- $\beta$ , the simple root corresponding to the maximal parabolic  $\mathbf{P}$ ;
- $\xi$ , the fundamental weight corresponding to the maximal parabolic  $\mathbf{P}$ ;
- $\rho = \rho_{\mathbf{G}} = \sum_{i \in D_{\mathbf{G}}} \omega_i \in P_{\mathbf{G}}$ ;
- $\rho_{\mathbf{H}_a} = \sum_{i \in D_a} \omega_i \in P_{\mathbf{H}_a}$ ;
- $(-, -)$ , the scalar product on the root/weight lattices;

#### (3) Weyl groups

- $\mathbf{W}_{\mathbf{G}}$ ,  $\mathbf{W}_{\mathbf{L}}$ ,  $\mathbf{W}_{\mathbf{L}_{\text{inn}}}$ ,  $\mathbf{W}_{\mathbf{L}_{\text{out}}}$ ,  $\mathbf{W}_{\mathbf{H}_a}$ , the Weyl groups;

- $s_\alpha, s_i = s_{\alpha_i}, s_\beta$ , the simple reflections in the simple roots;
  - $\ell : \mathbf{W} \rightarrow \mathbb{Z}_{\geq 0}$ , the length function on a Weyl group;
  - $w_0^{\mathbf{G}}, w_0^{\mathbf{L}}, w_0^{\mathbf{L}_{\text{inn}}}, w_0^{\mathbf{L}_{\text{out}}}, w_0^{\mathbf{H}_a}$ , the longest elements in the corresponding Weyl groups;
  - $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$ , the set of special representatives of left  $\mathbf{W}_{\mathbf{L}}$ -cosets in  $\mathbf{W}_{\mathbf{G}}$ , see section 2.5;
  - $\text{SR}_{\mathbf{H}}^{\mathbf{M}}$ , the set of special representatives of left  $\mathbf{W}_{\mathbf{M}}$ -cosets in  $\mathbf{W}_{\mathbf{H}}$ ;
- (4) Maps
- $i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{L}, \mathbf{L}_{\text{inn}} \rightarrow \mathbf{H}_a, \mathbf{L}_{\text{inn}} \rightarrow \mathbf{G}$ , the natural embeddings;
  - $o : \mathbf{L}_{\text{out}} \rightarrow \mathbf{L}, \mathbf{L}_{\text{out}} \rightarrow \mathbf{G}$ , the natural embeddings;
  - $h_a : \mathbf{H}_a \rightarrow \mathbf{G}$ , the natural embedding;
  - $i^* : P_{\mathbf{G}} \rightarrow P_{\mathbf{L}_{\text{inn}}}, o^* : P_{\mathbf{G}} \rightarrow P_{\mathbf{L}_{\text{out}}}, h_a^* : P_{\mathbf{G}} \rightarrow P_{\mathbf{H}_a}$ , the restriction of weights;
  - $i_* : Q_{\mathbf{L}_{\text{inn}}} \rightarrow Q_{\mathbf{G}}, o_* : Q_{\mathbf{L}_{\text{out}}} \rightarrow Q_{\mathbf{G}}, h_{a*} : Q_{\mathbf{H}_a} \rightarrow Q_{\mathbf{G}}$ , the embedding of roots;
- (5) Representations and bundles
- $V_{\mathbf{G}}^\lambda$ , the irreducible representation of  $\mathbf{G}$  with highest weight  $\lambda \in P_{\mathbf{G}}^+$ ;
  - $V_{\mathbf{L}}^\lambda$ , the irreducible representation of  $\mathbf{L}$  with highest weight  $\lambda \in P_{\mathbf{L}}^+$ ;
  - $\mathcal{U}^\lambda$ , the  $\mathbf{G}$ -equivariant vector bundle on  $\mathbf{G}/\mathbf{P}$  corresponding to  $(V_{\mathbf{L}}^\lambda)_{|\mathbf{P}}$ ;
- (6) Other
- $X = \mathbf{G}/\mathbf{P}$ , the Grassmannian;
  - $\mathcal{D}(X)$ , the derived category of coherent sheaves on  $X$ ;
  - $\mathcal{D}^{\mathbf{G}}(X)$ , the derived category of  $\mathbf{G}$ -equivariant coherent sheaves on  $X$ ;

**2.2. Roots and weights.** Let  $\mathbf{G}$  be a simple algebraic group,  $\mathbf{P}$  be the maximal parabolic subgroup,  $\mathbf{G}/\mathbf{P} = X$ . Let  $\beta$  be the corresponding simple root of  $\mathbf{G}$  and  $\xi$  the corresponding fundamental weight.

We denote by  $\mathbf{U} \subset \mathbf{P}$  the unipotent radical of  $\mathbf{P}$  and by  $\mathbf{L} = \mathbf{P}/\mathbf{U}$  the Levi quotient. Recall that the projection  $\mathbf{P} \rightarrow \mathbf{L}$  admits a splitting. Thus,  $\mathbf{L}$  can be considered as a subgroup of  $\mathbf{P}$ , and hence of  $\mathbf{G}$ . Note also that the set of simple roots of  $\mathbf{L}$  is the complement of  $\beta$  in the set of simple roots of  $\mathbf{G}$ .

The embedding of groups  $\mathbf{L} \subset \mathbf{G}$  induces an isomorphism of weight lattices  $P := P_{\mathbf{G}} \xrightarrow{\sim} P_{\mathbf{L}}$ . We use this isomorphism to identify the lattices. Let  $P_{\mathbf{L}}^+$  and  $P_{\mathbf{G}}^+$  denote the dominant cones in  $P$  of  $\mathbf{L}$  and  $\mathbf{G}$  respectively.

We identify simple roots of the group  $\mathbf{G}$  with the vertices of the Dynkin diagram  $D_{\mathbf{G}}$ . In particular, we say that simple roots  $\alpha$  and  $\alpha'$  are adjacent if the corresponding vertices are connected by an edge, or equivalently if  $\alpha \neq \alpha'$  but  $(\alpha, \alpha') \neq 0$ .

The fundamental weight of  $\mathbf{G}$  corresponding to the vertex  $i \in D_{\mathbf{G}}$  is denoted by  $\omega_i$ . Also, we denote by  $\rho = \rho_{\mathbf{G}}$  half the sum of simple roots of  $\mathbf{G}$ , or equivalently, the sum of fundamental weights.

We consider the root lattice  $Q_{\mathbf{G}}$  of  $\mathbf{G}$  as a sublattice of the weight lattice (roots are weights in the adjoint representation). We denote by  $(-, -)$  the scalar product on the weight–root lattices. This scalar product is defined uniquely up to a multiplicative constant. We choose the standard scaling as in [Bou]. Note that with this choice all scalar products of roots are integer and scalar products of weights are rational.

**2.3. Weyl group action.** The simple reflection at root  $\alpha = \alpha_i$  is denoted by  $s_\alpha = s_{\alpha_i} = s_i$ . Note that

$$(2) \quad s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_j,$$

which means that

$$(3) \quad (\omega_j, \alpha_i) = \delta_{ij}\alpha_i^2/2.$$

It follows that

$$(4) \quad s_i\rho = \rho - \alpha_i$$

for all  $i$ .

We identify  $\mathbf{W}_{\mathbf{L}}$  with the subgroup in  $\mathbf{W}_{\mathbf{G}}$  generated by all simple reflections  $s_{\alpha_i}$  with  $\alpha_i \neq \beta$ . Together with (2) this immediately implies the following

**Lemma 2.1.** *The weight  $\xi$  is invariant under the action of  $\mathbf{W}_{\mathbf{L}}$ .*

The length function on the Weyl group is denoted by  $\ell$  (recall that  $\ell(w)$  is the length of a minimal representation of  $w$  as a product of simple reflections). The following Lemma is well-known (see [Hum], Lemma 10.3A and its proof).

**Lemma 2.2.** *If  $w \in \mathbf{W}_{\mathbf{G}}$  and  $s_j$  is a simple reflection corresponding to the simple root  $\alpha_j$  then one has  $\ell(ws_j) > \ell(w)$  if and only if the root  $w(\alpha_j)$  is positive.*

Recall that the dominant cone  $P_{\mathbf{G}}^+$  is a fundamental domain for the action of  $\mathbf{W}_{\mathbf{G}}$  on  $P_{\mathbf{G}}$ . In particular, for each  $\lambda \in P_{\mathbf{G}}$  there is an element  $w \in \mathbf{W}_{\mathbf{G}}$  such that  $w\lambda \in P_{\mathbf{G}}^+$ . Moreover, such  $w$  is unique unless  $\lambda$  is orthogonal to a root of  $\mathbf{G}$  (in the other words, unless  $\lambda$  lies on a wall of a Weyl chamber).

Denote by  $Q_{\mathbf{G}}^+ \subset P_{\mathbf{G}}$  the cone of all linear combinations of simple roots with nonnegative integer coefficients. The following is also well known but we provide a proof for completeness.

**Lemma 2.3.** *If  $\lambda$  is dominant then for any  $w \in \mathbf{W}_{\mathbf{G}}$  we have  $\lambda - w\lambda \in Q_{\mathbf{G}}^+$ .*

*Proof.* Since  $\lambda$  is a positive linear combination of fundamental weights, it is enough to check that for every  $\omega_i$  and every  $w$  the weight  $\omega_i - w\omega_i$  is a sum of positive roots. This can be checked by induction in the length of  $w$ . When  $w$  is a simple reflection  $s_j$  this follows from (2). Let  $s = s_j$  be a simple reflection, and assume  $\ell(ws_j) = \ell(w) + 1$ . Then

$$\omega_i - ws_j\omega_i = \omega_i - w\omega_i + w(\omega_i - s_j\omega_i) = \omega_i - w\omega_i + w(\delta_{ij}\alpha_j).$$

Now the assertion follows from the induction assumption and from Lemma 2.2.  $\square$

The following consequence of this Lemma will be extremely important for us.

**Corollary 2.4.** *For a pair of weights  $\lambda$  and  $\mu$  the maximum (resp. minimum) of the scalar product  $(w\lambda, \mu)$  is achieved when  $w\lambda$  and  $\mu$  lie in the same (resp. opposite) Weyl chambers.*

*Proof.* Since the scalar product is  $\mathbf{W}$ -invariant we can assume that  $\mu$  is dominant. Then we have to check that if  $\lambda$  is also dominant then  $(w\lambda, \mu) \leq (\lambda, \mu)$  for any  $w \in \mathbf{W}$ . But this follows easily from Lemma 2.3 and nonnegativity of the scalar products of positive roots and dominant weights. The claim about the minimum follows as well since the minimum of  $(w\lambda, \mu)$  is minus the maximum of  $(w\lambda, -\mu)$ .  $\square$

Assume that  $\mathbf{G}$  is a simply connected semisimple algebraic group and let  $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_k$  be its decomposition into the product of simple groups. Then  $P_{\mathbf{G}} = P_{\mathbf{G}_1} \oplus \cdots \oplus P_{\mathbf{G}_k}$  and  $P_{\mathbf{G}}^+ = P_{\mathbf{G}_1}^+ \times \cdots \times P_{\mathbf{G}_k}^+$ . Denote by  $\lambda_i$  the component of a weight  $\lambda \in P_{\mathbf{G}}$  in the summand  $P_{\mathbf{G}_i}$ .

**Definition 2.5.** A weight  $\lambda \in P_{\mathbf{G}}^+$  is strictly dominant if all its components  $\lambda_i \in P_{\mathbf{G}_i}$  are distinct from 0.

**Lemma 2.6.** *If  $\lambda, \mu \in P_{\mathbf{G}}^+$  then  $(\lambda, \mu) \geq 0$ . Moreover, if  $\lambda$  is strictly dominant and  $\mu \neq 0$  then  $(\lambda, \mu) > 0$ .*

*Proof.* Follows immediately from the fact that all scalar products of fundamental weights of a simple group are strictly positive.  $\square$

Let  $\beta$  be the simple root corresponding to  $\mathbf{P}$ . The  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$  has the following nice description.

**Lemma 2.7.** *The  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$  consists of all roots of  $\mathbf{G}$  that have the coefficient of  $\beta$  equal to 1 and have the same length as  $\beta$ .*

*Proof.* The coefficient of  $\beta$  in a root  $\alpha$  is given by  $(\xi, \alpha)/(\xi, \beta)$ , where  $\xi$  is the fundamental weight corresponding to  $\beta$ . Since  $\xi$  is invariant under the action of  $\mathbf{W}_{\mathbf{L}}$  we have

$$(\xi, w_{\mathbf{L}}\beta) = (w_{\mathbf{L}}^{-1}\xi, \beta) = (\xi, \beta)$$

for all  $w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}$ , so the coefficient of  $\beta$  is equal to 1 for all roots in the  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$ .

Conversely, let us check that if a positive root  $\alpha$  has the coefficient of  $\beta$  equal to 1 and  $(\alpha, \alpha) = (\beta, \beta)$  then  $\alpha$  is in the  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$ . Let us write  $\alpha = \sum c_i \alpha_i$ , where  $\alpha_i$  are simple roots. We will use induction in  $\sum c_i$ . If  $\sum c_i = 1$  then  $\alpha = \beta$ , so the statement is true. Now assume that  $\sum c_i > 1$ . It is enough to prove that there exists a simple root  $\alpha_i \neq \beta$  such that  $(\alpha, \alpha_i) > 0$ . Indeed, then  $s_i \alpha$  will have smaller sum of coefficients and by the induction assumption we would deduce that  $s_i \alpha$  is in  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$ . Suppose  $(\alpha, \alpha_i) \leq 0$  for all  $\alpha_i \neq \beta$ . Then

$$(\alpha, \alpha) = (\beta + \sum_{\alpha_i \neq \beta} c_i \alpha_i, \alpha) = (\beta, \alpha) + \sum_{\alpha_i \neq \beta} c_i (\alpha_i, \alpha) \leq (\beta, \alpha).$$

Since  $(\alpha, \alpha) = (\beta, \beta)$  by assumption we get  $(\beta, \beta) \leq (\beta, \alpha)$ . But  $s_{\beta}(\alpha) = \alpha - 2\frac{(\alpha, \beta)}{(\beta, \beta)}\beta$  should be a positive root (since  $\alpha \neq \beta$ ). Looking at the coefficient of  $\beta$  in  $s_{\beta}(\alpha)$  we obtain

$$2\frac{(\alpha, \beta)}{(\beta, \beta)} \leq 1$$

which contradicts the previous inequality.  $\square$

We denote by  $w_0^{\mathbf{G}}$  and  $w_0^{\mathbf{L}}$  the longest elements of the Weyl groups  $\mathbf{W}_{\mathbf{G}}$  and  $\mathbf{W}_{\mathbf{L}}$  respectively. Note that

$$(w_0^{\mathbf{L}})^2 = (w_0^{\mathbf{G}})^2 = 1.$$

Note also that  $w_0^{\mathbf{G}}$  takes any simple root of  $\mathbf{G}$  to minus simple root, and hence any fundamental weight to minus fundamental weight. In particular,

$$(5) \quad w_0^{\mathbf{G}} \rho_{\mathbf{G}} = -\rho_{\mathbf{G}}$$

and  $w_0^{\mathbf{G}}(P_{\mathbf{G}}^+) = -P_{\mathbf{G}}^+$ ,  $w_0^{\mathbf{L}}(P_{\mathbf{L}}^+) = -P_{\mathbf{L}}^+$ .

**2.4. Representations.** For each dominant weight  $\lambda \in P_{\mathbf{G}}^+$  (resp.  $\lambda \in P_{\mathbf{L}}^+$ ) we denote by  $V_{\mathbf{G}}^{\lambda}$  (resp.  $V_{\mathbf{L}}^{\lambda}$ ) the corresponding irreducible representation of  $\mathbf{G}$  (resp.  $\mathbf{L}$ ).

The dual of any irreducible representation is also irreducible. To be more precise we have

$$(6) \quad (V_{\mathbf{L}}^{\lambda})^{\vee} = V_{\mathbf{L}}^{-w_0^{\mathbf{L}} \cdot \lambda}.$$

Indeed, if  $\lambda$  is the highest weight of an irreducible representation of  $\mathbf{L}$  then  $w_0^{\mathbf{L}}\lambda$  is the lowest weight, so  $-w_0^{\mathbf{L}}\lambda$  is the highest weight of the dual.

Since the group  $\mathbf{L}$  is reductive the tensor product of two irreducible representations of  $\mathbf{L}$  is a direct sum of irreducibles. We denote by  $\text{mult}(V_{\mathbf{L}}^{\nu}, V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu})$  the multiplicity of  $V_{\mathbf{L}}^{\nu}$  in the tensor product. The following simple result will be useful.

**Lemma 2.8.** *We have*

$$\text{mult}(V_{\mathbf{L}}^{\nu}, V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu}) = \dim \text{Hom}(V_{\mathbf{L}}^{\nu}, V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu}) = \dim \text{Hom}(V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu}, V_{\mathbf{L}}^{\nu}).$$

*In particular,*  $\text{mult}(V_{\mathbf{L}}^{\nu}, V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu}) = \text{mult}((V_{\mathbf{L}}^{\mu})^{\vee}, V_{\mathbf{L}}^{\lambda} \otimes (V_{\mathbf{L}}^{\nu})^{\vee})$ .

*Proof.* The first part follows from the fact that there are no maps between different irreducibles and just one map between isomorphic irreducibles. The second part follows from the canonical isomorphism  $\text{Hom}(V_{\mathbf{L}}^{\nu}, V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{\mu}) \cong \text{Hom}((V_{\mathbf{L}}^{\mu})^{\vee}, V_{\mathbf{L}}^{\lambda} \otimes (V_{\mathbf{L}}^{\nu})^{\vee})$ .  $\square$

We also need the following standard result restricting the highest weights of irreducible summands of a tensor product (obtained e.g. by combining [FH, Thm. 14.18] with [Zhel, §131, Thm. 5]; see also [Hum, Exer. 24.12] and [FH, Exer. 25.33]).

**Lemma 2.9.** *If  $\text{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) > 0$  then  $\nu \in \text{Conv}(\lambda + w\mu)_{w \in \mathbf{W}_{\mathbf{L}}}$ , where  $\text{Conv}$  stands for the convex hull and  $\mathbf{W}_{\mathbf{L}}$  is the Weyl group of  $\mathbf{L}$ . Similarly, if  $\text{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes (V_{\mathbf{L}}^\mu)^\vee) > 0$  then  $\nu \in \text{Conv}(\lambda - w\mu)_{w \in \mathbf{W}_{\mathbf{L}}}$ .*

**2.5. Special representatives.** In each coset  $\mathbf{W}_{\mathbf{L}}w \subset \mathbf{W}$  by  $\mathbf{W}_{\mathbf{L}}$  there is a unique representative which takes the  $\mathbf{G}$ -dominant cone to the  $\mathbf{L}$ -dominant cone. We call it the  $\mathbf{L}$ -special representative of the coset and denote the set of all  $\mathbf{L}$ -special representatives in  $\mathbf{W}$  by  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . Note that the cardinality of  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  equals to the cardinality of  $\mathbf{W}_{\mathbf{G}}/\mathbf{W}_{\mathbf{L}}$ , that is to the rank of the Grothendieck group  $K_0(X)$ .

The elements of  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  can also be characterized as follows.

**Lemma 2.10.** *The set  $\text{SR}_{\mathbf{G}}^{\mathbf{L}} \subset \mathbf{W}$  consists of the elements that have minimal length in their left  $\mathbf{W}_{\mathbf{L}}$ -cosets.*

*Proof.* Let  $w \in \mathbf{W}$  be an element of minimal length in its left  $\mathbf{W}_{\mathbf{L}}$ -coset. Then  $\ell(w^{-1}s_j) = \ell(s_jw) > \ell(w) = \ell(w^{-1})$  for every simple reflection  $s_j$  in  $\mathbf{W}_{\mathbf{L}}$ . Hence, by Lemma 2.2, the root  $w^{-1}(\alpha_j)$  is positive for every simple root  $\alpha_j$  that belongs to the root system of  $\mathbf{L}$ . Thus,  $(w\rho, \alpha_j) = (\rho, w^{-1}\alpha_j) > 0$  for every such simple root, i.e.,  $w\rho$  is  $\mathbf{L}$ -dominant. In other words,  $w$  takes  $P_{\mathbf{G}}^+$  to  $P_{\mathbf{L}}^+$ , so it is a special representative.  $\square$

**Lemma 2.11.** (0) *The only element of length 0 in  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  is 1.*

(1) *The only element of length 1 in  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  is  $s_\beta$ .*

(2) *All elements of length 2 in  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  are equal to  $s_\beta s_\alpha$ , where  $\alpha$  is the simple root of  $\mathbf{G}$  adjacent to  $\beta$ .*

*Proof.* The first is clear. For the second we note that elements of length 1 in  $\mathbf{W}_{\mathbf{G}}$  are just simple reflections and for  $\alpha \neq \beta$  the reflection  $s_\alpha$  is in the same  $\mathbf{W}_{\mathbf{L}}$  coset as 1 which has smaller length. Similarly, all elements of length 2 are products  $s_{\alpha_1}s_{\alpha_2}$  of simple reflections. If  $\alpha_1 \neq \beta$  then  $s_{\alpha_2}$  is in the same coset and has smaller length, hence  $\alpha_1 = \beta$ . And if  $\alpha := \alpha_2$  is not adjacent to  $\beta$  then reflections  $s_\alpha$  and  $s_\beta$  commute, so  $s_\beta s_\alpha = s_\alpha s_\beta$  is in the same coset as  $s_\beta$  which has smaller length.  $\square$

Take any reductive subgroup  $\mathbf{H} \subset \mathbf{G}$  compatible with the maximal torus in  $\mathbf{G}$ , and let  $\mathbf{M} = \mathbf{H} \cap \mathbf{L}$  be the Levi of  $\mathbf{H}$ . Let  $\mathbf{W}_{\mathbf{H}}$  and  $\mathbf{W}_{\mathbf{M}}$  be corresponding Weyl groups. Note that  $\mathbf{W}_{\mathbf{M}} = \mathbf{W}_{\mathbf{H}} \cap \mathbf{W}_{\mathbf{L}}$ . It follows that  $\mathbf{W}_{\mathbf{H}}/\mathbf{W}_{\mathbf{M}} \subset \mathbf{W}_{\mathbf{G}}/\mathbf{W}_{\mathbf{L}}$ . Actually, the same inclusion holds for the sets of special representatives.

**Lemma 2.12.** *We have  $\text{SR}_{\mathbf{G}}^{\mathbf{L}} \cap \mathbf{W}_{\mathbf{H}} = \text{SR}_{\mathbf{H}}^{\mathbf{M}}$ .*

*Proof.* Let  $w \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$ . We have to show that  $(w\rho, \alpha_i) \geq 0$ , where  $\alpha_i$  is any simple root of  $\mathbf{L}$ . If  $\alpha_i$  belongs to the root system of  $\mathbf{H} \cap \mathbf{L} = \mathbf{M}$  then this follows from the definition of  $\text{SR}_{\mathbf{H}}^{\mathbf{M}}$ . Otherwise,  $\ell(w^{-1}s_i) > \ell(w^{-1})$ , where  $s_i$  is the simple reflection associated with  $\alpha_i$ . Hence, by Lemma 2.2,  $w^{-1}\alpha_i$  is a positive root, and so  $(w\rho, \alpha_i) = (\rho, w^{-1}\alpha_i) \geq 0$ .  $\square$

**Lemma 2.13.** *Assume that  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . Then*

$$(\xi, \rho - v\rho) \geq \ell(v)(\xi, \beta),$$

*where  $\beta$  is the simple root corresponding to  $\xi$ . If  $\ell(v) = 1$  then the above is an equality.*

*Proof.* Let us prove this by induction in the length of  $v$ . In the case  $v = 1$  both sides of our inequality are equal to zero. Now assume that  $\ell(v) \geq 1$ . Recall that  $v$  is the representative of minimal length in the coset  $\mathbf{W}_{\mathbf{L}}v$ . Thus, we can write  $v = us_i$ , where  $s_i$  is a simple reflection,  $\ell(u) = \ell(v) - 1$  and  $u \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . We have

$$(\xi, \rho - v\rho) = (\xi, \rho - u\rho) + (\xi, u(\rho - s_i\rho)) = (\xi, \rho - u\rho) + (\xi, u(\alpha_i)).$$

The first summand in the right-hand side is  $\geq \ell(u)(\xi, \beta)$  by the induction assumption. Thus, it suffices to check that  $(\xi, u(\alpha_i)) \geq (\xi, \beta)$ .

Since  $\ell(us_i) = \ell(u) + 1$ , the root  $u(\alpha_i)$  is positive (by Lemma 2.2), so we only have to check that  $\beta$  appears in  $u(\alpha_i)$  with nonzero coefficient, i.e., that  $(\xi, u(\alpha_i)) \neq 0$ . Suppose  $(\xi, u(\alpha_i)) = 0$ . Then  $u(\alpha_i) = \sum_{\alpha_j \neq \beta} n_j \alpha_j$  with  $n_j \geq 0$ . The fact that  $v$  has minimal length in its right  $W_{\mathbf{L}}$ -cosets implies that  $\ell(v^{-1}s_j) > \ell(v^{-1})$  for every  $j$  such that  $\alpha_j \neq \beta$ . Hence, all the roots  $v^{-1}(\alpha_j)$  are positive, and therefore,

$$-\alpha_i = s_i \alpha_i = v^{-1} u(\alpha_i) = \sum_{\alpha_j \neq \beta} n_j v^{-1}(\alpha_j)$$

should be positive, so we get a contradiction.

If  $\ell(v) = 1$  then  $v = s_\beta$  by Lemma 2.11 hence  $\rho - v\rho = \beta$  and both sides are equal to  $(\xi, \beta)$ .  $\square$

*Remark 2.14.* Note that if the root  $\beta$  is **cominuscul**, which means that the coefficient of  $\beta$  in any root of  $\mathbf{G}$  does not exceed 1, then

$$(\xi, \rho - v\rho) = \ell(v)(\xi, \beta)$$

for all  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . Indeed, in the argument above we conclude that the coefficient of  $\beta$  in  $u(\alpha_i)$  is precisely 1, hence we obtain an inductive proof of the equality.

**2.6. Equivariant bundles  $\mathcal{U}^\lambda$  and Borel–Bott–Weil Theorem.** Since  $X = \mathbf{G}/\mathbf{P}$  is a homogeneous variety, the category  $\text{Coh}^{\mathbf{G}}(X)$  of  $\mathbf{G}$ -equivariant coherent sheaves on  $X$  is equivalent to the category of representations of  $\mathbf{P}$ :

$$(7) \quad \text{Coh}^{\mathbf{G}}(X) \cong \text{Rep } -\mathbf{P}.$$

This equivalence is compatible with the structures of tensor abelian categories on both sides, i.e. it preserves tensor products and duals.

For each  $\lambda \in P_{\mathbf{L}}^+$ , a dominant weight of the Levi quotient  $\mathbf{L} = \mathbf{P}/\mathbf{U}$ , we consider  $V_{\mathbf{L}}^\lambda$ , the corresponding irreducible representation of  $\mathbf{L}$ . Restricting  $V_{\mathbf{L}}^\lambda$  to  $\mathbf{P}$  (via the projection  $\mathbf{P} \rightarrow \mathbf{L}$ ) we obtain a representation of  $\mathbf{P}$ , and hence a  $\mathbf{G}$ -equivariant vector bundle on  $X$  which we denote by  $\mathcal{U}^\lambda$ . Since the above equivalence preserves the tensor structure we deduce from Lemma 2.8 and (6) that

$$(8) \quad \mathcal{U}^\lambda \otimes \mathcal{U}^{\lambda'} = \bigoplus_{\mu \in P_{\mathbf{L}}^+} \text{Hom}(V_{\mathbf{L}}^\mu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^{\lambda'}) \otimes \mathcal{U}^\mu, \quad (\mathcal{U}^\lambda)^\vee \cong \mathcal{U}^{-w_0^{\mathbf{L}}\lambda}.$$

Note that  $V_{\mathbf{L}}^\xi$  is a one-dimensional representation of  $\mathbf{L}$ , hence  $\mathcal{U}^\xi$  is a line bundle on  $X$ . Moreover, it is the ample generator of  $\text{Pic } X = \mathbb{Z}$ , so we will denote it by  $\mathcal{O}_X(1)$ . Thus,

$$(9) \quad \mathcal{O}_X(t) = \mathcal{U}^{t\xi}.$$

Similarly, we will denote the bundle  $\mathcal{U}^{\lambda+t\xi}$  by  $\mathcal{U}^\lambda(t)$ .

The cohomology groups of bundles  $\mathcal{U}^\lambda$  can be computed via the Borel–Bott–Weil Theorem. Recall that a weight  $\lambda \in P_{\mathbf{G}}$  is called **G-singular** if it lies on a wall of a Weyl chamber of  $\mathbf{G}$  (equivalently, if it is orthogonal to some root of  $\mathbf{G}$ ). If a weight does not lie on a wall of a Weyl chamber it is called **G-regular**. The sets of singular and regular weights are invariant under the natural action of the Weyl group  $\mathbf{W}_{\mathbf{G}}$  on  $P_{\mathbf{G}}$ .

**Theorem 2.15.** ([Bott, Thm. IV']) *Take any  $\lambda \in P_{\mathbf{L}}^+ \subset P_{\mathbf{L}} = P_{\mathbf{G}}$ . If  $\lambda + \rho_{\mathbf{G}}$  is **G-singular** then  $H^\bullet(X, \mathcal{U}^\lambda) = 0$ . If  $\lambda + \rho_{\mathbf{G}}$  is **G-regular** then there exists a unique  $w \in \mathbf{W}_{\mathbf{G}}$  such that  $w(\lambda + \rho_{\mathbf{G}})$  is strictly dominant. In this case*

$$H^{\ell(w)}(X, \mathcal{U}^\lambda) = V_{\mathbf{G}}^{w \cdot (\lambda + \rho_{\mathbf{G}}) - \rho_{\mathbf{G}}}$$

*and other cohomology groups vanish. In particular, if  $\lambda$  is **G-dominant** then  $H^0(X, \mathcal{U}^\lambda) = V_{\mathbf{G}}^\lambda$ .*

Let  $P_{\mathbf{G}}^{\text{reg}}$  denote the set of all regular weights of  $\mathbf{G}$  and  $P_{\mathbf{G}}^{\text{reg}} - \rho_{\mathbf{G}}$  denote the set of all weights  $\mu \in P_{\mathbf{G}}$  such that  $\mu + \rho_{\mathbf{G}} \in P_{\mathbf{G}}^{\text{reg}}$ . Further, for each  $\mu \in P_{\mathbf{G}}^{\text{reg}} - \rho_{\mathbf{G}}$  denote by  $w_{\mu}$  the unique element of the Weyl group  $\mathbf{W}_{\mathbf{G}}$  such that  $w_{\mu}(\mu + \rho_{\mathbf{G}})$  is  $\mathbf{G}$ -dominant. Combining above Theorem with (8) we deduce

**Corollary 2.16.** *We have*

$$\text{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\lambda'}) = \bigoplus_{\mu \in \text{Conv}(\lambda' - w\lambda)_{w \in \mathbf{W}_{\mathbf{L}}} \cap P_{\mathbf{L}}^{\text{reg}} - \rho_{\mathbf{G}}} \text{Hom}(V_{\mathbf{L}}^{\mu}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \otimes V_{\mathbf{G}}^{w_{\mu}(\mu + \rho_{\mathbf{G}}) - \rho_{\mathbf{G}}}[-\ell(w_{\mu})],$$

where  $[-\ell(w_{\mu})]$  stands for cohomological shift.

We will also need a way to compute Ext-groups in the equivariant derived category  $\mathcal{D}^{\mathbf{G}}(X)$ . Denote those Ext groups between  $F, F' \in \mathcal{D}^{\mathbf{G}}(X)$  by  $\text{Ext}_{\mathbf{G}}^i(F, F') = \text{Hom}_{\mathcal{D}^{\mathbf{G}}(X)}(F, F'[i])$ .

**Proposition 2.17.** *We have*

- (i)  $\text{Ext}_{\mathbf{G}}^i(F, F') = (\text{Ext}^i(F, F'))^{\mathbf{G}}$ , the space of  $\mathbf{G}$ -invariants in the Ext-group between  $F$  and  $F'$  in  $\mathcal{D}(X)$ .
- (ii)  $\text{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\lambda'}) = \bigoplus_{v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}} \text{Hom}(V_{\mathbf{L}}^{v\rho - \rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})[-\ell(v)]$ .
- (iii)  $\text{Ext}_{\mathbf{G}}^1(\mathcal{U}^{\lambda}, \mathcal{U}^{\lambda'}) = \text{Hom}(V_{\mathbf{L}}^{-\beta}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})$ .

*Proof.* The first claim follows from  $\text{Hom}_{\mathbf{G}}(F, F') = \text{Hom}(F, F')^{\mathbf{G}}$  because the functor of invariants is exact (since the group  $\mathbf{G}$  is reductive).

For the second note that  $(V_{\mathbf{G}}^{\nu})^{\mathbf{G}}$  is zero for  $\nu \neq 0$  and  $\mathbf{k}$  for  $\nu = 0$ , hence  $\mu$  from the formula of Corollary 2.16 contributes to  $\text{Ext}_{\mathbf{G}}$  if and only if  $w_{\mu}(\mu + \rho) - \rho = 0$ , that is if  $\mu = v\rho - \rho$  for some  $v \in \mathbf{W}_{\mathbf{G}}$ . Since  $\mu$  should be  $\mathbf{L}$ -dominant, the element  $v$  should be a special representative, that is  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . Of course, if  $v\rho - \rho \notin \text{Conv}(\lambda' - w\lambda)$  then  $\text{Hom}$  is zero, so we can easily forget this restriction.

Finally, to get the third claim we use the fact that by Lemma 2.11(1) the only special representative of length 1 is  $s_{\beta}$  and  $s_{\beta}\rho = \rho - \beta$ .  $\square$

**2.7. The canonical class.** The canonical class of the Grassmannian  $\mathbf{G}/\mathbf{P}$  is well known to be isomorphic to the line bundle  $\mathcal{U}^{w_0^{\mathbf{L}}w_0^{\mathbf{G}}\rho - \rho}$ . We will need the following more precise formula.

**Lemma 2.18.** *Let  $\beta$  be the simple root corresponding to  $\mathbf{P}$  and  $\xi$  the corresponding fundamental weight. Let  $\bar{\beta}$  be the maximal root in  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$ . Let*

$$r = (\rho, \bar{\beta} + \beta)/(\xi, \beta).$$

*Then  $\omega_X = \mathcal{U}^{-r\xi}$ .*

*Proof.* The Picard group of  $\mathbf{G}/\mathbf{P}$  is generated by  $\mathcal{U}^{\xi}$ , hence  $\omega_X \cong \mathcal{U}^{w_0^{\mathbf{L}}w_0^{\mathbf{G}}\rho - \rho} \cong \mathcal{U}^{-k\xi}$  for some  $k \in \mathbb{Z}$ . To find  $k$  we compute the scalar product with  $\beta$ . We get

$$k = (\rho - w_0^{\mathbf{L}}w_0^{\mathbf{G}}\rho, \beta)/(\xi, \beta).$$

Further  $(-w_0^{\mathbf{L}}w_0^{\mathbf{G}}\rho, \beta) = (w_0^{\mathbf{L}}\rho, \beta) = (\rho, w_0^{\mathbf{L}}\beta)$  by (5). Now note that  $\beta$  considered as a weight of  $\mathbf{L}$  is antidominant (its scalar products with the simple roots of  $\mathbf{L}$  are nonpositive), hence  $w_0^{\mathbf{L}}\beta$  is  $\mathbf{L}$ -dominant. Moreover, it is a positive root since  $(\xi, w_0^{\mathbf{L}}\beta) = (w_0^{\mathbf{L}}\xi, \beta) = (\xi, \beta) > 0$  since  $\xi$  is  $\mathbf{W}_{\mathbf{L}}$ -invariant. Hence,  $\bar{\beta} := w_0^{\mathbf{L}}\beta$  is the maximal root in the  $\mathbf{W}_{\mathbf{L}}$ -orbit of  $\beta$  by Lemma 2.3 and the claim follows.  $\square$

*Remark 2.19.* By Lemma 2.7,  $\bar{\beta}$  is in fact the maximal root of the same length as  $\beta$  and with the coefficient of  $\beta$  equal to 1. This gives a very easy way to find  $\bar{\beta}$  just by looking into the table of roots.

*Remark 2.20.* The integer  $r$  is called the index of the Grassmannian  $\mathbf{G}/\mathbf{P}$ .

The following consequence of the above formula is useful.

**Corollary 2.21.** *Let  $\mathbf{P}$  be a maximal parabolic subgroup in  $\mathbf{G}$  and  $\beta$  the corresponding simple root. Let  $\mathbf{H} \subset \mathbf{H}' \subset \mathbf{G}$  be a pair of semisimple subgroups corresponding to a pair of Dynkin subdiagrams  $D_{\mathbf{H}} \subset D_{\mathbf{H}'} \subset D_{\mathbf{G}}$  such that  $\beta \in D_{\mathbf{H}}$  and there is a simple root  $\alpha \in D_{\mathbf{H}'} \setminus D_{\mathbf{H}}$  adjacent to the connected component of  $\beta$  in  $D_{\mathbf{H}}$ . Let  $r$  and  $r'$  be the indices of the Grassmannians  $\mathbf{H}/(\mathbf{H} \cap \mathbf{P})$  and  $\mathbf{H}'/(\mathbf{H}' \cap \mathbf{P})$  respectively. Then  $r' > r$ .*

*Proof.* Let  $\mathbf{M} = \mathbf{L} \cap \mathbf{H}$  and  $\mathbf{M}' = \mathbf{L} \cap \mathbf{H}'$ . Let  $\bar{\beta}$  be the maximal root in  $\mathbf{W}_{\mathbf{M}}$ -orbit of  $\beta$  and  $\bar{\beta}'$  the maximal root in  $\mathbf{W}_{\mathbf{M}'}$ -orbit of  $\beta$ . Let  $\alpha$  be any simple root of  $\mathbf{H}'$  adjacent to the connected component of  $\beta$  in  $D_{\mathbf{H}}$ . Note that since  $\bar{\beta}$  is maximal, the coefficient of any simple root of the connected component of  $\beta$  in  $D_{\mathbf{H}}$  in  $\bar{\beta}$  is strictly positive. In particular, the coefficients of roots adjacent to  $\alpha$  are positive, hence the scalar product  $(\alpha, \bar{\beta})$  is strictly negative. Therefore,

$$s_{\alpha}(\bar{\beta}) = \bar{\beta} - 2 \frac{(\alpha, \bar{\beta})}{\alpha^2} \alpha$$

has a strictly positive coefficient of  $\alpha$ . Therefore,  $(\rho, \bar{\beta}') \geq (\rho, s_{\alpha}(\bar{\beta})) \geq (\rho, \bar{\beta}) + (\rho, \alpha) > (\rho, \bar{\beta})$  since  $(\rho, \alpha) = \alpha^2/2 > 0$ .  $\square$

### 3. EXCEPTIONAL BLOCKS

Let  $\mathbf{G}$  be a simple simplyconnected algebraic group and  $\mathbf{P} \subset \mathbf{G}$  a maximal parabolic subgroup. We take  $X = \mathbf{G}/\mathbf{P}$  and denote by  $\mathcal{D}(X)$  the bounded derived category of coherent sheaves on  $X$  and  $\mathcal{D}^{\mathbf{G}}(X)$  — the bounded derived category of  $\mathbf{G}$ -equivariant coherent sheaves. We denote by  $\text{Fg} : \mathcal{D}^{\mathbf{G}}(X) \rightarrow \mathcal{D}(X)$  the forgetful functor.

We denote as usual  $\text{Ext}^i(F, F') = \text{Hom}(F, F'[i])$ , Ext-groups in category  $\mathcal{D}(X)$ . Similarly, Ext-groups in the equivariant category  $\mathcal{D}^{\mathbf{G}}(X)$  are denoted by  $\text{Ext}_{\mathbf{G}}^i(F, F')$ . Note that the forgetful functor induces a linear map

$$\text{Fg} : \text{Ext}_{\mathbf{G}}^i(F, F') \rightarrow \text{Ext}^i(F, F').$$

For each triple of  $\mathbf{L}$ -dominant weights  $\lambda, \mu, \nu \in P_{\mathbf{L}}^+$  consider the map

$$\text{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) \otimes \text{Hom}(\mathcal{U}^{\nu}, \mathcal{U}^{\mu}) \rightarrow \text{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\mu}),$$

the composition of the action of the forgetful functor with the Yoneda multiplication.

Now we can introduce the main notion of this section.

**Definition 3.1.** A set of  $\mathbf{L}$ -dominant weights  $\mathbf{B} \subset P_{\mathbf{L}}^+$  is called an exceptional block if for all  $\lambda, \mu \in \mathbf{B}$  the canonical map

$$(10) \quad \bigoplus_{\nu \in \mathbf{B}} \text{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) \otimes \text{Hom}(\mathcal{U}^{\nu}, \mathcal{U}^{\mu}) \rightarrow \text{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\mu})$$

is an isomorphism.

The goal of this section is to show that for any exceptional block  $\mathbf{B} \subset P_{\mathbf{L}}^+$  the category

$$\mathcal{D}_{\mathbf{B}}(X) = \langle \mathcal{U}^{\lambda} \rangle_{\lambda \in \mathbf{B}} \subset \mathcal{D}(X)$$

generated in  $\mathcal{D}(X)$  by bundles  $\mathcal{U}^{\lambda}$  with  $\lambda \in \mathbf{B}$ , has a full exceptional collection.

**3.1. The  $\xi$ -ordering.** Recall that  $\beta$  is the simple root of  $\mathbf{G}$  corresponding to the maximal parabolic  $\mathbf{P}$  and  $\xi$  is the corresponding fundamental weight. By Lemma 2.1 it is invariant under the action of  $\mathbf{W}_{\mathbf{L}}$ .

Consider the partial ordering on the weight lattice  $P_{\mathbf{L}}$  defined by:

$$(11) \quad \begin{array}{ll} \lambda \prec \mu & \text{if } (\xi, \lambda) < (\xi, \mu) \\ \lambda \preceq \mu & \text{if either } \lambda \prec \mu \text{ or } \lambda = \mu \end{array}$$

We will call it the  $\xi$ -ordering.



**Lemma 3.2.** *If  $\text{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$  then  $\lambda \preceq \mu$ .*

*Proof.* By Corollary 2.16 if  $\text{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$  then there is a non-trivial  $\mathbf{G}$ -map  $V_{\mathbf{L}}^\kappa \subset (V_{\mathbf{L}}^\lambda)^\vee \otimes V_{\mathbf{L}}^\mu$  for some  $\mathbf{G}$ -dominant weight  $\kappa$ . This means that there is a non-trivial  $\mathbf{G}$ -map  $V_{\mathbf{L}}^\kappa \otimes V_{\mathbf{L}}^\lambda \rightarrow V_{\mathbf{L}}^\mu$ , hence  $\mu \in \text{Conv}(\lambda + w\kappa)_{w \in \mathbf{W}_{\mathbf{L}}}$  by Lemma 2.9. But for any  $w \in \mathbf{W}_{\mathbf{L}}$  we have

$$(\xi, \lambda + w\kappa) - (\xi, \lambda) = (\xi, w\kappa) = (w^{-1}\xi, \kappa) = (\xi, \kappa) \geq 0,$$

the last inequality follows from Lemma 2.6 since both  $\xi$  and  $\kappa$  are  $\mathbf{G}$ -dominant. Moreover, since  $\mathbf{G}$  is simple the weight  $\xi$  is strictly dominant, hence the last inequality is strict unless  $\kappa = 0$ . Thus, we see that  $\lambda \prec \mu$  unless  $\kappa = 0$ . But if  $\kappa = 0$  then  $V_{\mathbf{L}}^\kappa \otimes V_{\mathbf{L}}^\lambda = V_{\mathbf{L}}^\lambda$ , hence  $\mu = \lambda$ .  $\square$

Thus, we see that  $\text{Hom}$  groups between  $\mathcal{U}^\lambda$  in  $\mathcal{D}(X)$  go in the direction of the  $\xi$ -ordering. It turns out that  $\text{Ext}$  groups in the equivariant category go in the opposite direction!

**Lemma 3.3.** *If  $\text{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$  then  $\mu \preceq \lambda$ . More precisely, if  $\text{Ext}_{\mathbf{G}}^i(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$  then*

$$(\xi, \lambda) - (\xi, \mu) \geq i(\xi, \beta),$$

*and for  $i = 1$  this inequality becomes an equality. Also, each bundle  $\mathcal{U}^\lambda$  is exceptional in  $\mathcal{D}^{\mathbf{G}}(X)$ .*

*Proof.* By Proposition 2.17 if  $\text{Ext}_{\mathbf{G}}^i(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$  then there is a non-trivial  $\mathbf{G}$ -map  $V_{\mathbf{L}}^{v\rho-\rho} \rightarrow (V_{\mathbf{L}}^\lambda)^\vee \otimes V_{\mathbf{L}}^\mu$  for some  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$  with  $\ell(v) = i$ . This means that there is a non-trivial  $\mathbf{G}$ -map  $V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^\lambda \rightarrow V_{\mathbf{L}}^\mu$ , hence  $\mu \in \text{Conv}(\lambda + w(v\rho - \rho))_{w \in \mathbf{W}_{\mathbf{L}}}$  by Lemma 2.9. Now by Lemma 2.13, for any  $w \in \mathbf{W}_{\mathbf{L}}$  we have

$$(\xi, \lambda + w(v\rho - \rho)) - (\xi, \lambda) = (\xi, w(v\rho - \rho)) = (w^{-1}\xi, (v\rho - \rho)) = (\xi, v\rho - \rho) \leq -i(\xi, \beta),$$

where the last inequality becomes an equality for  $i = 1$ . This implies that

$$(\xi, \mu) - (\xi, \lambda) \leq -i(\xi, \beta)$$

with equality for  $i = 1$ , as required. Thus, we see that  $\mu \prec \lambda$  unless  $v = 1$ . But if  $v = 1$  then  $V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^\lambda = V_{\mathbf{L}}^\lambda$ , hence  $\mu = \lambda$ . Also, if  $v = 1$  then  $i = \ell(v) = 0$ , so  $\text{Ext}_{\mathbf{G}}^{>0}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = 0$  and by Proposition 2.17 we have  $\text{Hom}_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = \text{Hom}(V_{\mathbf{L}}^\lambda, V_{\mathbf{L}}^\lambda) = \mathbf{k}$ , hence  $\mathcal{U}^\lambda$  is exceptional in  $\mathcal{D}^{\mathbf{G}}(X)$ .  $\square$

The last Lemma has the following interesting consequence.

**Theorem 3.4.** *The bundles  $\{\mathcal{U}^\lambda\}_{\lambda \in P_{\mathbf{L}}^+}$  ordered with respect to any total ordering extending the opposite of the  $\xi$ -ordering constitute a full exceptional collection in the equivariant category  $\mathcal{D}^{\mathbf{G}}(X)$ .*

*Proof.* The fact that we get an exceptional collection follows from Lemma 3.3. It remains to check that it is full.

Indeed, let us show that every object is generated by this collection. It suffices to check this only for pure objects, that is for  $\mathbf{G}$ -equivariant coherent sheaves. As we know they can be considered as just  $\mathbf{P}$ -representations. But each representation of  $\mathbf{P}$  has a filtration (an extension of the radical filtration) all the quotients of which are simple  $\mathbf{L}$ -representations, i.e. correspond to bundles  $\mathcal{U}^\lambda$  with appropriate  $\lambda \in P_{\mathbf{L}}^+$ . Thus, it is contained in the subcategory generated by  $\mathcal{U}^\lambda$ .  $\square$

*Remark 3.5.* The fact that the orderings of  $\text{Hom}$ 's in  $\mathcal{D}(X)$  and  $\text{Ext}$ 's in  $\mathcal{D}^{\mathbf{G}}(X)$  are opposite is the reason for the fact that  $\mathcal{U}^\lambda$  are not exceptional in  $\mathcal{D}(X)$  — one can construct a nontrivial self- $\text{Ext}$  of  $\mathcal{U}^\lambda$  in  $\mathcal{D}(X)$  by composing  $\text{Hom}$ 's and equivariant  $\text{Ext}$ 's. As we will see in section 3.3 below, the cure is to reverse in a sense one of the orderings.

**3.2. The forgetful functor and its adjoint.** Let  $B \subset P_L^+$  be an exceptional block. Let

$$\mathcal{D}_B^G(X) = \langle \mathcal{U}^\lambda \rangle_{\lambda \in B}$$

denote the subcategory of  $\mathcal{D}^G(X)$  generated by  $\mathcal{U}^\lambda$  with  $\lambda$  in  $B$ . Since the collection  $\{\mathcal{U}^\lambda\}_{\lambda \in B}$  is exceptional, the category  $\mathcal{D}_B^G$  is saturated (see [BV]), hence the forgetful functor  $Fg : \mathcal{D}_B^G(X) \rightarrow \mathcal{D}_B(X)$  has a right adjoint functor which we denote by  $Fg^! : \mathcal{D}_B(X) \rightarrow \mathcal{D}_B^G(X)$ .

The crucial observation is the following

**Proposition 3.6.** *If  $B$  is an exceptional block then*

$$Fg^!(Fg(\mathcal{U}^\mu)) = \bigoplus_{\nu \in B} \text{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu,$$

where  $\text{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu)$  are considered just as vector spaces, not as representations of  $G$ .

*Proof.* Let

$$\tilde{\mathcal{U}}^\mu := \bigoplus_{\nu \in B} \text{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu \in \mathcal{D}_B^G(X).$$

We have a canonical evaluation map  $\text{ev} : Fg(\tilde{\mathcal{U}}^\mu) \rightarrow Fg(\mathcal{U}^\mu)$  in  $\mathcal{D}(X)$ . By adjunction it gives a map  $\tilde{\mathcal{U}}^\mu \rightarrow Fg^!Fg(\mathcal{U}^\mu)$ . Let us show it is an isomorphism. For this let us check that the induced map

$$f : \text{Ext}_G^\bullet(\mathcal{U}^\lambda, \tilde{\mathcal{U}}^\mu) \rightarrow \text{Ext}_G^\bullet(\mathcal{U}^\lambda, Fg^!Fg(\mathcal{U}^\mu))$$

is an isomorphism for all  $\lambda \in B$ . Indeed, we have a commutative diagram

$$\begin{array}{ccccc} \bigoplus_{\nu \in B} \text{Ext}_G^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \text{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) & \xlongequal{\quad} & \text{Ext}_G^\bullet(\mathcal{U}^\lambda, \tilde{\mathcal{U}}^\mu) & \xrightarrow{f} & \text{Ext}_G^\bullet(\mathcal{U}^\lambda, Fg^!Fg(\mathcal{U}^\mu)) \\ \downarrow Fg \otimes 1 & & \downarrow Fg & & \parallel \\ \bigoplus_{\nu \in B} \text{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \text{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) & \xlongequal{\quad} & \text{Ext}^\bullet(Fg(\mathcal{U}^\lambda), Fg(\tilde{\mathcal{U}}^\mu)) & \xrightarrow{\text{ev}} & \text{Ext}^\bullet(Fg(\mathcal{U}^\lambda), Fg(\mathcal{U}^\mu)) \end{array}$$

The composition of the left vertical map with the maps in the bottom row is the map of (10) which is an isomorphism since  $B$  is an exceptional block. Hence, the map  $f$  in the top row is an isomorphism as well.

It follows that the cone of the map  $\tilde{\mathcal{U}}^\mu \rightarrow Fg^!Fg(\mathcal{U}^\mu)$  is orthogonal to all  $\mathcal{U}^\lambda$  in  $\mathcal{D}_B^G(X)$ . But  $\mathcal{U}^\lambda$  generate this category, hence the cone is zero.  $\square$

*Question 3.7.* It is interesting to find a general formula for  $Fg^!$  (or maybe for  $Fg^! \circ Fg$ ).

**3.3. Exceptional bundles  $\mathcal{E}^\lambda$ .** The crucial step is to replace the exceptional collection  $\mathcal{U}^\lambda$  in  $\mathcal{D}_B^G(X)$  by its right dual exceptional collection (see [B]).

Recall that if  $(E, F)$  is an exceptional pair in a triangulated category  $\mathcal{T}$  then the right mutation  $\mathbf{R}_F(E)$  is defined as the cone

$$\mathbf{R}_F(E) := \text{Cone}(E \xrightarrow{\text{coev}} \text{Hom}^\bullet(E, F)^\vee \otimes F),$$

It is well known that  $(F, \mathbf{R}_F(E))$  is also an exceptional pair which generates the same subcategory in  $\mathcal{T}$  as the initial pair  $(E, F)$ .

Now assume that  $E_1, \dots, E_n$  is an exceptional collection. Its right dual collection is defined as the collection obtained by a sequence of right mutations:

$$(E_n, \mathbf{R}_{E_n} E_{n-1}[-1], \mathbf{R}_{E_n} \mathbf{R}_{E_{n-1}} E_{n-2}[-2], \dots, \mathbf{R}_{E_n} \cdots \mathbf{R}_{E_2} E_1[-n+1]).$$

This collection is exceptional and generates the same subcategory as the initial collection.

Now we apply this construction to the exceptional collection  $(\mathcal{U}^\lambda)_{\lambda \in \mathbf{B}}$  in the equivariant derived category  $\mathcal{D}^{\mathbf{G}}(X)$  and denote by

$$(12) \quad \mathcal{E}_{\mathbf{B}}^\lambda := \mathbf{R}_{\langle \mathcal{U}^\mu \rangle_{\{\mu \in \mathbf{B} \mid \mu \prec \lambda\}}} \mathcal{U}^\lambda,$$

the objects of the right dual collection. Further on we will frequently drop the index  $\mathbf{B}$  in the notation  $\mathcal{E}_{\mathbf{B}}^\lambda$  if it is clear which block  $\mathbf{B}$  is considered.

By definition the objects  $\mathcal{E}^\lambda$  are exceptional in the equivariant derived category. Our goal now is to show that the objects  $\mathbf{Fg}(\mathcal{E}^\lambda)$  in the usual derived category  $\mathcal{D}(X)$  are also exceptional and moreover form a full exceptional collection in  $\mathcal{D}_{\mathbf{B}}(X)$ .

First of all, recall that the standard property of the right dual exceptional collections gives

$$(13) \quad \mathrm{Ext}_{\mathbf{G}}^\bullet(\mathcal{E}^\lambda, \mathcal{U}^\mu) = \begin{cases} \mathbf{k}, & \text{for } \lambda = \mu \\ 0, & \text{otherwise} \end{cases}$$

(see e.g. [B]). Also, it follows from the construction of the dual collection that the subcategories both in  $\mathcal{D}^{\mathbf{G}}(X)$  and  $\mathcal{D}(X)$  generated by objects  $\mathcal{E}^\mu$  and  $\mathcal{U}^\mu$  coincide:

$$(14) \quad \langle \mathcal{E}^\mu \rangle_{\mu \preceq \lambda} = \langle \mathcal{U}^\mu \rangle_{\mu \preceq \lambda},$$

and moreover, for each  $\lambda$  there is a morphism  $\mathcal{E}^\lambda \rightarrow \mathcal{U}^\lambda$  such that

$$(15) \quad \mathrm{Cone}(\mathcal{E}^\lambda \rightarrow \mathcal{U}^\lambda) \in \langle \mathcal{U}^\mu \rangle_{\mu \prec \lambda}.$$

**Corollary 3.8.** *For all  $\lambda, \mu \in \mathbf{B}$  we have*

$$(16) \quad \mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) = \mathrm{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu).$$

*Proof.* Indeed, by Proposition 3.6 we have

$$\mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) \cong \mathrm{Ext}_{\mathbf{G}}^\bullet(\mathcal{E}^\lambda, \mathbf{Fg}^!(\mathcal{U}^\mu)) \cong \mathrm{Ext}_{\mathbf{G}}^\bullet(\mathcal{E}^\lambda, \bigoplus_{\nu \in \mathbf{B}} \mathrm{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu).$$

Now note that by (13) we have  $\mathrm{Ext}_{\mathbf{G}}^\bullet(\mathcal{E}^\lambda, \mathcal{U}^\nu) = 0$  unless  $\lambda = \nu$ . Thus, the RHS equals  $\mathrm{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu)$ .  $\square$

**Proposition 3.9.** *For an exceptional block  $\mathbf{B}$  the objects  $\mathbf{Fg}(\mathcal{E}^\lambda)$  form a full exceptional collection in  $\mathcal{D}_{\mathbf{B}}(X)$ . The ordering of the collection is the  $\xi$ -ordering.*

*Proof.* First, take  $\mu \prec \lambda$ . By (16) and Lemma 3.2 we have  $\mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) = 0$ . Then (14) implies  $\mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathbf{Fg}(\mathcal{E}^\mu)) = 0$  as well. On the other hand, using this semiorthogonality and (15) we deduce that  $\mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathbf{Fg}(\mathcal{E}^\lambda)) \cong \mathrm{Ext}^\bullet(\mathbf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\lambda) = \mathrm{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = \mathbf{k}$ , so each  $\mathbf{Fg}(\mathcal{E}^\lambda)$  is exceptional. Finally, the fullness of the collection  $\{\mathbf{Fg}(\mathcal{E}^\lambda)\}_{\lambda \in \mathbf{B}}$  in  $\mathcal{D}_{\mathbf{B}}(X)$  follows from (14).  $\square$

From now on to unburden the notation we will denote  $\mathbf{Fg}(\mathcal{E}^\lambda)$  simply by  $\mathcal{E}^\lambda$ .

**3.4. Properties of exceptional blocks.** Let  $\mathbf{B}$  be any subset of  $P_{\mathbf{L}}^+$  and  $\mu \in P_{\mathbf{L}}^+$ . Denote

$$\mathbf{B} + \mu = \{\lambda + \mu \mid \lambda \in \mathbf{B}\}.$$

**Lemma 3.10.** *If  $\mathbf{B}$  is an exceptional block then for each  $t \in \mathbb{Z}$  the block  $\mathbf{B} + t\xi$  is exceptional. Moreover,  $\mathcal{E}_{\mathbf{B}+t\xi}^{\lambda+t\xi} = \mathcal{E}_{\mathbf{B}}^\lambda(t)$ .*

*Proof.* Recall that  $\mathcal{U}^{t\xi} = \mathcal{O}_X(t)$  and twisting by this bundle takes  $\mathcal{U}^\lambda$  to  $\mathcal{U}^{\lambda+t\xi}$ . Since such a twisting is an autoequivalence it follows that it preserves exceptionality of a block.  $\square$

Let us say that a subset  $\mathbf{B}' \subset \mathbf{B}$  is closed with respect to decreasing in  $\xi$ -ordering, if for any  $\lambda, \mu \in \mathbf{B}$  if  $\lambda \in \mathbf{B}'$  and  $\mu \preceq \lambda$  then  $\mu \in \mathbf{B}'$ .

**Lemma 3.11.** *Let  $B$  be an exceptional block and  $B' \subset B$  be a subset closed with respect to decreasing in  $\xi$ -ordering. Then  $B'$  is an exceptional block. Moreover,  $\mathcal{E}_{B'}^\lambda = \mathcal{E}_B^\lambda$  for all  $\lambda \in B'$ .*

*Proof.* Take  $\lambda, \mu \in B'$  and consider the map (10). It is an isomorphism since  $B$  is exceptional. On the other hand,  $\nu \in B$  contributes to the LHS only if  $\text{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\nu) \neq 0$  which by Lemma 3.3 implies that  $\nu \prec \lambda$ . But then  $\nu \in B'$  since  $B'$  is closed with respect to decreasing in  $\xi$ -ordering. Thus, the LHS of (10) coincides with the LHS of analogous map written for the block  $B'$ , hence  $B'$  is exceptional.

An isomorphism between  $\mathcal{E}_{B'}^\lambda$  and  $\mathcal{E}_B^\lambda$  follows immediately from the definition (12).  $\square$

**3.5. The output set and the criterion of exceptionality.** In this section we give a criterion for a block  $B$  to be exceptional in terms of the Weyl group action on weights and the representation theory of  $\mathbf{L}$ . We start with some preparations.

**Lemma 3.12.** *Let  $\mu \in P_{\mathbf{L}}^+ \cap (P_{\mathbf{G}}^{\text{reg}} - \rho)$ . Then there exists a unique pair  $(\kappa, v)$ , where  $\kappa \in P_{\mathbf{G}}^+$  and  $v \in \mathbf{W}_{\mathbf{G}}$  such that*

$$\mu = v(\kappa + \rho) - \rho.$$

*Moreover,  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ .*

*Proof.* Existence and uniqueness of the pair  $(\kappa, v)$  follow from regularity of  $\mu + \rho$ . And since  $\mu \in P_{\mathbf{L}}^+$  we conclude that  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ .  $\square$

Using this simple observation we can rewrite the formula of Corollary 2.16 as follows:

$$\text{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{\kappa \in P_{\mathbf{G}}^+, v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}} \mid v(\kappa + \rho) - \rho \in \text{Conv}(\lambda' - w\lambda)_{w \in \mathbf{W}_{\mathbf{L}}}} \text{Hom}(V_{\mathbf{L}}^{v(\kappa + \rho) - \rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \otimes V_{\mathbf{G}}^\kappa[-\ell(v)].$$

It is clear from this formula that it is convenient to have a control over the set of all pairs  $(\kappa, v)$  which can appear in the RHS. So, we define the **output set** for the pair of weights  $\lambda, \lambda'$  of  $\mathbf{L}$  as

$$\text{OP}(\lambda, \lambda') = \{(\kappa, v) \in P_{\mathbf{G}}^+ \times \text{SR}_{\mathbf{G}}^{\mathbf{L}} \mid v(\kappa + \rho) - \rho \in \text{Conv}(\lambda' - w\lambda)_{w \in \mathbf{W}_{\mathbf{L}}}\}.$$

Consequently, we define the **output set** of a block  $B$  to be

$$\text{OP}(B) = \bigcup_{\lambda, \lambda' \in B} \text{OP}(\lambda, \lambda') \subset P_{\mathbf{G}}^+ \times \text{SR}_{\mathbf{G}}^{\mathbf{L}},$$

and we denote by  $\text{OP}_1(B) \subset P_{\mathbf{G}}^+$  and  $\text{OP}_2(B) \subset \text{SR}_{\mathbf{G}}^{\mathbf{L}}$  the projections of  $\text{OP}(B)$  to  $P_{\mathbf{G}}^+$  and  $\text{SR}_{\mathbf{G}}^{\mathbf{L}}$  respectively, so that

$$\text{OP}(B) \subset \text{OP}_1(B) \times \text{OP}_2(B).$$

Using these definitions we can rewrite the formula of Corollary 2.16 as follows:

$$(17) \quad \text{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{(\kappa, v) \in \text{OP}(\lambda, \lambda')} \text{Hom}(V_{\mathbf{L}}^{v(\kappa + \rho) - \rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \otimes V_{\mathbf{G}}^\kappa[-\ell(v)].$$

Note that we can extend the area of summation to  $P_{\mathbf{G}}^+ \times \text{SR}_{\mathbf{G}}^{\mathbf{L}}$  in the above formula. Indeed, if for a pair  $(\kappa, v)$  one has  $v(\kappa + \rho) - \rho \notin \text{Conv}(\lambda' - w\lambda)_{w \in \mathbf{W}_{\mathbf{L}}}$  then  $\text{Hom}(V_{\mathbf{L}}^{v(\kappa + \rho) - \rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) = 0$  by Lemma 2.9, and we have no contribution. So, we can replace  $\text{OP}(\lambda, \lambda')$  by  $\text{OP}(B)$ , or even by  $\text{OP}_1(B) \times \text{OP}_2(B)$ .

Also, for each set of  $\mathbf{L}$ -dominant weights  $S \subset P_{\mathbf{L}}^+$  denote by  $\Pi_S : \text{Rep } \mathbf{L} \rightarrow \text{Rep } \mathbf{L}$  the projector onto the subcategory formed by all  $V_{\mathbf{L}}^\nu$  with  $\nu \in S$ . In other words,  $\Pi_S$  is a functor such that

$$\Pi_S(V_{\mathbf{L}}^\lambda) = \begin{cases} V_{\mathbf{L}}^\lambda, & \text{if } \lambda \in S \\ 0, & \text{otherwise} \end{cases}$$

**Proposition 3.13.** *Assume that a subset  $B \subset P_{\mathbf{L}}^+$  has the following two properties:*

- (a) for all  $\kappa \in \text{OP}_1(\mathbf{B})$ ,  $v \in \text{OP}_2(\mathbf{B})$  we have  $v\kappa = \kappa$ ;
- (b) for all  $\kappa \in \text{OP}_1(\mathbf{B})$ ,  $v \in \text{OP}_2(\mathbf{B})$ ,  $\lambda \in \mathbf{B}$  the canonical map

$$(18) \quad \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa+v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}) \rightarrow \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))$$

is an isomorphism.

Then the block  $\mathbf{B}$  is exceptional.

In what follows we will refer to part (a) of this criterion as **invariance condition**, and to part (b) as **compatibility condition**.

*Proof.* Fix a pair of weights  $\lambda, \lambda' \in \mathbf{B}$ . We have to check that the map (10) (with  $\mu = \lambda'$ ) is an isomorphism.

We start by rewriting (17) in a more convenient form. First of all, we extend the summation area to  $\text{OP}_1(\mathbf{B}) \times \text{OP}_2(\mathbf{B})$  (as was mentioned above, this does not spoil the equality). Next, we use the isomorphism

$$\text{Hom}(V_{\mathbf{L}}^{v(\kappa+\rho)-\rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \simeq \text{Hom}(V_{\mathbf{L}}^{\mu}, V_{\mathbf{L}}^{v(\kappa+\rho)-\rho} \otimes V_{\mathbf{L}}^{\lambda})^{\vee} \simeq \text{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v(\kappa+\rho)-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee},$$

where the second isomorphism follows from the condition  $\lambda' \in \mathbf{B}$ . Finally, by the invariance condition we have  $v(\kappa + \rho) - \rho = \kappa + v\rho - \rho$ . Thus, we obtain

$$(19) \quad \text{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\lambda'}) = \bigoplus_{\kappa \in \text{OP}_1(\mathbf{B}), v \in \text{OP}_2(\mathbf{B})} \text{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa+v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(v)],$$

Now specializing (19) we can obtain an expression for  $\text{Ext}_{\mathbf{G}}$  and  $\text{Hom}$  in the LHS of (10). To obtain an expression for  $\text{Ext}_{\mathbf{G}}$  we should restrict to the case  $\kappa = 0$ . Replacing also  $\lambda'$  by  $\nu \in \mathbf{B}$  we obtain

$$(20) \quad \text{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) = \bigoplus_{v \in \text{OP}_2(\mathbf{B})} \text{Hom}(V_{\mathbf{L}}^{\nu}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} [-\ell(v)].$$

On the other hand, to obtain an expression for  $\text{Hom}$  we should restrict to  $v = 1$ . Replacing also  $\lambda$  by  $\nu$  we obtain

$$(21) \quad \text{Hom}^{\bullet}(\mathcal{U}^{\nu}, \mathcal{U}^{\lambda'}) = \bigoplus_{\kappa \in \text{OP}_1(\mathbf{B})} \text{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{\nu}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}.$$

Combining (20) with (21) we rewrite the LHS of (10) as

$$\begin{aligned} \bigoplus_{\nu \in \mathbf{B}} \text{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) \otimes \text{Hom}^{\bullet}(\mathcal{U}^{\nu}, \mathcal{U}^{\lambda'}) = \\ \bigoplus_{\nu \in \mathbf{B}, \kappa \in \text{OP}_1(\mathbf{B}), v \in \text{OP}_2(\mathbf{B})} \text{Hom}(V_{\mathbf{L}}^{\nu}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} \otimes \text{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{\nu}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(v)] = \\ \bigoplus_{\kappa \in \text{OP}_1(\mathbf{B}), v \in \text{OP}_2(\mathbf{B})} \text{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda})))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(v)], \end{aligned}$$

where the second equality follows from the formula

$$\Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}) = \bigoplus_{\nu \in \mathbf{B}} \text{Hom}(V_{\mathbf{L}}^{\nu}, \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} \otimes V_{\mathbf{L}}^{\nu}.$$

To conclude we compare the obtained expression for the LHS of (10) with the expression (19) for the RHS and note that the map from the LHS of (10) to the RHS is induced by the map (18). Thus, if the compatibility property (b) holds then the map is an isomorphism, hence the block  $\mathbf{B}$  is exceptional.  $\square$

#### 4. ON STRONGNESS AND PURITY

Note that a priori the exceptional objects  $\mathcal{E}^\lambda$  constructed above are complexes. However, we have the following

**Conjecture 4.1.** *For any exceptional block  $\mathbf{B} \subset P_{\mathbf{L}}^+$  and any  $\lambda \in \mathbf{B}$  the object  $\mathcal{E}^\lambda$  is a vector bundle.*

Note that the standard  $t$ -structure on  $\mathcal{D}^{\mathbf{G}}(X)$  restricts to a  $t$ -structure on the category  $\mathcal{D}_{\mathbf{B}}^{\mathbf{G}}(X)$  whose core is  $\mathcal{C}_{\mathbf{B}}$  consists of  $\mathbf{G}$ -equivariant coherent sheaves that are obtained by successive extensions from  $\mathcal{U}^\lambda$  with  $\lambda \in \mathbf{B}$ . As it was already mentioned the category of  $\mathbf{G}$ -equivariant coherent sheaves on  $X$  is equivalent to the category of finite-dimensional representations of  $\mathbf{P}$ , which in turn is equivalent to the category of finite-dimensional representations of a certain infinite quiver with relations  $(\mathcal{Q}, \mathcal{I})$  (see [Hille]). Recall that the vertices of  $\mathcal{Q}$  are in bijection with the set  $P_{\mathbf{L}}^+$  of dominant weights of  $\mathbf{L}$ , and there is an arrow  $\lambda \rightarrow \mu$  if and only if  $V_{\mathbf{L}}^\mu$  appears in  $V_{\mathbf{L}}^{-\beta} \otimes V_{\mathbf{L}}^\lambda$  (i.e., when there is a nontrivial  $\text{Ext}_{\mathbf{G}}^1(\mathcal{U}^\lambda, \mathcal{U}^\mu)$ ). Note that by Lemma 3.3 the quiver is directed (by the opposite of the  $\xi$ -ordering). The subcategory  $\mathcal{C}_{\mathbf{B}}$  corresponds to the subcategory of representations supported at the vertices  $\mathbf{B} \subset P_{\mathbf{L}}^+$ . Hence, it is equivalent to the category of finite-dimensional representations of a finite directed quiver with relations  $(\mathcal{Q}_{\mathbf{B}}, \mathcal{I}_{\mathbf{B}})$ , where  $\mathcal{Q}_{\mathbf{B}} \subset \mathcal{Q}$  is the full subquiver corresponding to the set of vertices  $\mathbf{B}$ .

**Proposition 4.2.** *The following conditions are equivalent:*

- (1) *Each  $\mathcal{E}^\lambda$  for  $\lambda \in \mathbf{B}$  is a vector bundle.*
- (2) *For each  $\lambda \in \mathbf{B}$ ,  $\mathcal{E}^\lambda$  is a projective cover of  $\mathcal{U}^\lambda$  in the category  $\mathcal{C}_{\mathbf{B}}$ .*
- (3) *The natural map  $\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) \rightarrow \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu)$  is an isomorphism for  $\lambda, \mu \in \mathbf{B}$ .*
- (4) *The exceptional collection  $(\mathcal{E}^\lambda)_{\lambda \in \mathbf{B}}$  is strong.*

*Proof.* (1) $\Rightarrow$ (2). If  $\mathcal{E}^\lambda$  are vector bundles then they belong to  $\mathcal{C}_{\mathbf{B}}$ . Furthermore, since  $\mathcal{C}_{\mathbf{B}}$  is a core of a  $t$ -structure of a full subcategory  $\mathcal{D}_{\mathbf{B}}^{\mathbf{G}}(X)$  of  $\mathcal{D}^{\mathbf{G}}(X)$  we have  $\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^1(\mathcal{E}^\lambda, \mathcal{U}^\mu) \simeq \text{Ext}_{\mathbf{G}}^1(\mathcal{E}^\lambda, \mathcal{U}^\mu) = 0$  for  $\lambda, \mu \in \mathbf{B}$ . This implies that  $\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^1(\mathcal{E}^\lambda, \mathcal{F}) = 0$  for any  $\mathcal{F}$  in  $\mathcal{C}_{\mathbf{B}}$ , i.e.,  $\mathcal{E}^\lambda$  is projective.

(2) $\Rightarrow$ (1). If  $\mathcal{E}^\lambda$  is a projective cover of  $\mathcal{U}^\lambda$  in  $\mathcal{C}_{\mathbf{B}}$  then  $\mathcal{E}^\lambda$  itself is an object of  $\mathcal{C}_{\mathbf{B}}$ , hence a successive extension of  $\mathcal{U}^\mu$  with  $\mu \in \mathbf{B}$ . In particular, it is a vector bundle on  $X$ .

(2) $\Rightarrow$ (3). Using (2) we can construct for any object  $\mathcal{F}$  in  $\mathcal{C}_{\mathbf{B}}$  a projective resolution consisting of direct sums of objects  $\mathcal{E}^\lambda$ . Computing  $\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{F}, \mathcal{U}^\mu)$  using such a resolution and using the isomorphisms

$$\text{Hom}_{\mathcal{C}_{\mathbf{B}}}(\mathcal{E}^\lambda, \mathcal{U}^\mu) \simeq \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{E}^\lambda, \mathcal{U}^\mu)$$

we derive that the map  $\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{F}, \mathcal{U}^\mu) \rightarrow \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{F}, \mathcal{U}^\mu)$  is an isomorphism.

(3) $\Rightarrow$ (4). By Lemma 3.3, we have

$$\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) = \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) = 0$$

unless  $(\xi, \mu) < (\xi, \lambda)$ . Since the category  $\mathcal{C}_{\mathbf{B}}$  is equivalent to the category of finite-dimensional representations of a finite-dimensional algebra, for each  $\lambda \in \mathbf{B}$  there exists a projective cover  $\mathcal{P}^\lambda \rightarrow \mathcal{U}^\lambda$  in  $\mathcal{C}_{\mathbf{B}}$ . Now the condition (3) implies that the natural maps

$$\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{P}^\lambda, \mathcal{U}^\mu) \rightarrow \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{P}^\lambda, \mathcal{U}^\mu) \quad \text{and}$$

$$\text{Ext}_{\mathcal{C}_{\mathbf{B}}}^\bullet(\mathcal{P}^\lambda, \mathcal{P}^\mu) \rightarrow \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{P}^\lambda, \mathcal{P}^\mu)$$

are isomorphisms. It follows that  $(\mathcal{P}^\lambda)$  form a strong exceptional sequence and that  $\mathcal{P}^\lambda \simeq \mathcal{E}^\lambda$  for each  $\lambda \in \mathbf{B}$ .

(4) $\Rightarrow$ (1). Choose any ordering of  $\mathcal{U}^\lambda$  compatible with the partial ordering  $\prec$ . Let  $\mathcal{U}_p$  denote the  $p$ -th object for this ordering. Let  $\mathcal{E}_p$  be the objects of the dual collection. Then for any  $\mathcal{F} \in \mathcal{D}_{\mathbf{B}}^{\mathbf{G}}(X)$  there is a spectral sequence  $\text{Ext}_{\mathbf{G}}^q(\mathcal{E}_p[p], \mathcal{F}) \otimes \mathcal{U}_p \Rightarrow \mathcal{F}$ . Applying to  $\mathcal{F} = \mathcal{E}^\lambda$  gives (1).  $\square$

Now we are going to suggest several criteria when the equivalent conditions of Proposition 4.2 hold.

**Proposition 4.3.** *Assume that the subquiver  $\mathcal{Q}_B \subset \mathcal{Q}$  contains entirely any path that starts and ends in  $\mathcal{Q}_B$ . Then the equivalent conditions of Proposition 4.2 hold.*

*Proof.* Recall that the projective cover of a simple object of a vertex  $\lambda$  is the representation of  $(\mathcal{Q}_B, \mathcal{I}_B)$  associating with a vertex  $\mu \in B$  the vector space generated by all paths in the quiver from the vertex  $\lambda$  to  $\mu$  (modulo the relations). The condition of the Proposition ensures that this representation is isomorphic to the restriction to  $\mathcal{Q}_B$  of the projective cover of the simple object of the vertex  $\mu$  in the category of representations of  $\mathcal{Q}$ . It follows that  $\text{Hom}$ 's from projective objects to simple objects in  $\mathcal{Q}_B$  are the same as in  $\mathcal{Q}$ , and moreover, the restrictions to  $\mathcal{Q}_B$  of projective resolutions of simple objects in  $\mathcal{Q}$  give projective resolutions in  $\mathcal{Q}_B$ . Combining all this we deduce that  $\text{Ext}$ 's between simple objects in  $\mathcal{C}_B$  are isomorphic to those in  $\mathcal{C} = \text{Coh}^G(X)$ , i.e. the condition (3) of Proposition 4.2 holds.  $\square$

Also, the properties of purity and strongness of the collection  $\mathcal{E}^\lambda$  are related to Koszulity of a certain algebra, see [PP].

**Proposition 4.4.** (i) *Assume that the graded algebra*

$$A_B = \bigoplus_{\lambda, \mu \in B} \text{Ext}_{\mathbf{G}}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu)$$

*is Koszul (with respect to the cohomological grading). Then the equivalent conditions of Proposition 4.2 hold.*

(ii) *If the algebra  $A_B$  is one-generated then Koszulity of  $A_B$  is equivalent to the conditions of Proposition 4.2.*

*Proof.* (i) This follows from the main result of [Pos, Cor. 8] (see also the proofs of Theorems 4.1 and 4.2 in [P97]).

(ii) If the condition (3) of Proposition 4.2 is satisfied then  $A_B$  is isomorphic (as a graded algebra) to the  $\text{Ext}$ -algebra between simple objects in the abelian category  $\mathcal{C}_B$ . Thus, the assumption that  $A_B$  is one-generated implies that  $\mathcal{C}_B$  is a Koszul category, and so the algebra  $A_B$  is Koszul.  $\square$

*Remark 4.5.* In the case when the unipotent radical of  $\mathbf{P}$  is abelian (in this case the Grassmannian  $X = \mathbf{G}/\mathbf{P}$  is called *cominuscule*) and the subquiver  $\mathcal{Q}_B \subset \mathcal{Q}$  contains entirely any path that starts and ends in  $\mathcal{Q}_B$ , the algebra  $A_B$  is Koszul as follows from the main result of [Hille] and from Proposition 4.3. Note that in this case the function  $\lambda \mapsto -(\xi, \lambda)/(\xi, \beta)$  is a Koszul weight function on simple objects of  $\mathcal{C}_B$ . Indeed, this follows from Remark 2.14 using the argument of Lemma 3.3.

*Remark 4.6.* In 9.3 we will give an example (Example 9.5) of an exceptional block for which Proposition 4.3 does not apply, and at the same time the inequality of Lemma 3.3 becomes strict in some cases (and so algebra  $A_B$  is not one-generated) and so Proposition 4.4 does not apply as well, but the equivalent conditions of Proposition 4.2 still hold.

## 5. CONSTRUCTING EXCEPTIONAL BLOCKS

In this section we suggest a construction of an exceptional block which depends on a choice of a semisimple subgroup  $\mathbf{H} \subset \mathbf{G}$ . We start with some preparation.

5.1. **Cores.** Let  $\mathbf{H}$  be a semisimple group. In this subsection we will omit the subscript  $\mathbf{H}$  to unburden the notation. Let  $\delta \in P_{\mathbf{H}}^+$  be a strictly dominant weight (see Definition 2.5).

**Definition 5.1.** The polyhedron

$$(22) \quad \mathbf{R}_{\delta} = \{\lambda \in P_{\mathbf{H}} \otimes \mathbb{R} \mid \forall w \in \mathbf{W}_{\mathbf{H}} (w\delta, \lambda) \leq (\delta, \rho_{\mathbf{H}})\}$$

is called the core of shape  $\delta$ .

We will denote by

$$(23) \quad \mathbf{R}_{\delta}^* := \{\lambda \in P_{\mathbf{H}} \otimes \mathbb{R} \mid \forall w \in \mathbf{W}_{\mathbf{H}} (w\delta, \lambda) < (\delta, \rho_{\mathbf{H}})\}.$$

the interior of the core  $\mathbf{R}_{\delta}$ . Note that both  $\mathbf{R}_{\delta}$  and  $\mathbf{R}_{\delta}^*$  are  $\mathbf{W}_{\mathbf{H}}$ -invariant and convex.

**Lemma 5.2.** *The intersection of a core with the set of dominant weights is given by*

$$\mathbf{R}_{\delta} \cap P_{\mathbf{H}}^+ = \{\lambda \in P_{\mathbf{H}}^+ \mid (\delta, \lambda) \leq (\delta, \rho_{\mathbf{H}})\}.$$

Similarly,

$$\mathbf{R}_{\delta}^* \cap P_{\mathbf{H}}^+ = \{\lambda \in P_{\mathbf{H}}^+ \mid (\delta, \lambda) < (\delta, \rho_{\mathbf{H}})\}.$$

*Proof.* Let us check the first equality (the second is proved analogously). By definition, the LHS is contained in the RHS. On the other hand, since both  $\lambda$  and  $\delta$  are  $\mathbf{H}$ -dominant, by Corollary 2.4, we have  $(w\delta, \lambda) \leq (\delta, \lambda)$  for all  $w \in \mathbf{W}_{\mathbf{H}}$ , hence the RHS is contained in the LHS.  $\square$

**Lemma 5.3.** *All integer points of  $\mathbf{R}_{\delta}^*$  are singular. All regular integer points of the core  $\mathbf{R}_{\delta}$  are contained in the  $\mathbf{W}_{\mathbf{H}}$ -orbit of  $\rho_{\mathbf{H}}$ .*

*Proof.* Assume that  $\lambda \in P_{\mathbf{H}} \cap \mathbf{R}_{\delta}$  is regular. Take  $w \in \mathbf{W}_{\mathbf{H}}$  such that  $w\lambda$  is  $\mathbf{H}$ -dominant. Then  $w\lambda \in \mathbf{R}_{\delta} \cap P_{\mathbf{H}}^+$  and since  $w\lambda$  is regular we can write  $w\lambda = \rho_{\mathbf{H}} + \mu$ ,  $\mu \in P_{\mathbf{H}}^+$ . Therefore,

$$(\delta, \rho_{\mathbf{H}} + \mu) = (\delta, w\lambda) = (w^{-1}\delta, \lambda) \leq (\delta, \rho_{\mathbf{H}}),$$

hence  $(\delta, \mu) \leq 0$ . Since  $\delta$  is strictly dominant, this implies  $\mu = 0$  by Lemma 2.6, hence  $\lambda = w^{-1}\rho_{\mathbf{H}}$ .  $\square$

5.2. **The setup.** Consider the complement  $D_{\mathbf{G}} \setminus \beta$  of the vertex  $\beta$  of the Dynkin diagram  $D_{\mathbf{G}}$  of  $\mathbf{G}$ . In general it consists of several (up to 3) connected components of different types. We choose one component of type  $A$  (possibly empty) to be called the outer component and denote it by  $D_{\text{out}}$ . The union of the others component will be called the inner component and denoted by  $D_{\text{inn}}$ . We denote the corresponding connected semisimple groups by  $\mathbf{L}_{\text{out}}$  and  $\mathbf{L}_{\text{inn}}$  and by

$$o : \mathbf{L}_{\text{out}} \rightarrow \mathbf{L}, \quad i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{L}$$

the canonical embeddings. Abusing the notation we will also denote by  $o$  (resp.,  $i$ ) the embedding of  $\mathbf{L}_{\text{out}}$  (resp.,  $\mathbf{L}_{\text{inn}}$ ) into  $\mathbf{G}$ . Note that the groups  $\mathbf{L}_{\text{out}}$  and  $\mathbf{L}_{\text{inn}}$  are simply connected (this follows from the fact that an embedding of Dynkin diagrams induces a surjection of the weight lattices). In particular, we have

$$(24) \quad \mathbf{L}_{\text{out}} \cong \text{SL}_k$$

for some  $k \geq 1$ . We fix a numbering of the vertices of  $D = D_{\mathbf{G}}$  as follows. First, we number the vertices of the outer part  $D_{\text{out}} = A_{k-1}$  by integers from 1 to  $k-1$  in a standard way. Then we number the vertex  $\beta$  by  $k$  and the remaining vertices in an arbitrary way. We denote by  $b$  the number of the vertex in  $D_{\text{out}}$  which is adjacent to  $\beta$  (note that such vertex is unique).

Note that we have the following decomposition of the Weyl group of  $\mathbf{L}$ :

$$\mathbf{W}_{\mathbf{L}} = \mathbf{W}_{\mathbf{L}_{\text{out}}} \times \mathbf{W}_{\mathbf{L}_{\text{inn}}}.$$



(since  $D_{\text{out}}$  and  $D_{\text{inn}}$  are not adjacent the corresponding simple reflections commute).

Now consider the chain of subdiagrams

$$D_b \subset D_{b-1} \subset \cdots \subset D_1 \subset D_0 = D_{\mathbf{G}}, \quad D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}.$$

Let

$$\mathbf{H}_b \subset \mathbf{H}_{b-1} \subset \cdots \subset \mathbf{H}_1 \subset \mathbf{H}_0 = \mathbf{G}$$

be the corresponding chain of semisimple subgroups of  $\mathbf{G}$ . For  $a = 0, \dots, b$  we denote by

$$h_a : \mathbf{H}_a \rightarrow \mathbf{G}$$

the embedding. Note that any  $\mathbf{H}_a$  contains  $\mathbf{L}_{\text{inn}}$ . Abusing the notation we will denote the corresponding embedding by  $i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{H}_a$ .

For each  $a = 0, \dots, b$  we choose a strictly dominant weight  $\delta_a \in P_{\mathbf{H}_a}^+$  and consider the corresponding core  $\mathbf{R}_{\delta_a} \subset P_{\mathbf{H}_a} \otimes \mathbb{R}$ . To unburden the notation we denote this core by  $\mathbf{R}_a$ . The interior of this core is denoted by  $\mathbf{R}_a^*$ .

Let  $r$  be the index of  $\mathbf{G}/\mathbf{P}$  and let  $r_a$  be the index of  $\mathbf{H}_a/(\mathbf{H}_a \cap \mathbf{P})$ . Note that by Corollary 2.21 we have

$$0 < r_b < r_{b-1} < \cdots < r_1 < r_0 = r$$

**5.3. The indexing set.** Denote by  $\theta$  an element of  $P_{\mathbf{L}} \otimes \mathbb{Q}$  such that

$$(25) \quad \theta \in \langle \omega_1, \dots, \omega_{k-1} \rangle^\perp \cap \text{Ker } i^*, \quad \text{and} \quad (\theta, \xi) = 1$$

(it is easy to see that such  $\theta$  always exists and is unique). Note that the set  $(\theta, P_{\mathbf{L}})$  of all scalar products of  $\theta$  with weights of  $\mathbf{L}$  is a cyclic subgroup of  $\mathbb{Q}$  containing  $\mathbb{Z}$ . We consider the intersection of this subgroup with half-closed interval  $[0, r) \subset \mathbb{Q}$ :

$$\mathbf{J} = \{j \in (\theta, P_{\mathbf{L}}) \mid 0 \leq j < r\}.$$

This set will number the blocks in the collection. Note that it is naturally linearly ordered. The blocks will be shown to be semiorthogonal with respect to this order.

For each  $j \in \mathbf{J}$  there is a unique integer  $a(j)$  in the interval  $0 \leq a(j) \leq b$  such that

$$(26) \quad r - r_{a(j)} \leq j < r - r_{a(j)+1},$$

where we set  $r_{b+1} = 0$ . To unburden the notation we will write  $\mathbf{H}_j = \mathbf{H}_{a(j)}$ ,  $h_j = h_{a(j)}$  and  $\mathbf{R}_j = \mathbf{R}_{\delta_{a(j)}}$ .

Below we will need the following simple observation

**Lemma 5.4.** *For any  $\nu \in P_{\mathbf{L}_{\text{inn}}}$  there is a rational number  $p \in (\theta, P_{\mathbf{L}})$  such that  $p\xi + i_*\nu \in P_{\mathbf{L}}$ .*

*Proof.* Since any  $\nu$  is a linear combination of fundamental weights it suffices to consider the case of  $\nu = i^*\omega_t$  for some  $t \in D_{\text{inn}}$ . Then it is clear that  $i_*\nu = i_*i^*\omega_t$  is just the orthogonal projection of  $\omega_t$  onto the subspace  $i_*(P_{\mathbf{L}_{\text{inn}}} \otimes \mathbb{Q}) \subset P_{\mathbf{L}} \otimes \mathbb{Q}$ . Its orthogonal complement is spanned by  $\alpha \in Q_{\mathbf{L}_{\text{out}}}$  and by  $\xi$ . Moreover,  $\omega_t$  is orthogonal to all such  $\alpha$  since  $t \in D_{\text{inn}}$ . Hence,

$$i_*i^*\omega_t = \omega_t - \frac{(\omega_t, \xi)}{\xi^2} \xi.$$

It remains to check that  $(\omega_t, \xi)/\xi^2 \in (\theta, P_{\mathbf{L}})$ . For this we apply the linear function  $(\theta, -)$  to the above equality. Since  $\theta$  is orthogonal to the image of  $i_*$  we conclude that  $(\omega_t, \xi)/\xi^2 = (\theta, \omega_t) \in (\theta, P_{\mathbf{L}})$ .  $\square$

**5.4. The first approximation.** For each element of the indexing set  $j \in J$  we will define a subset  $\hat{B}_j \subset P_{\mathbf{L}}^+$ . We will show that this is an exceptional block unless  $\mathbf{G}$  is of type  $A$ . In the latter case we will have to replace  $\hat{B}_j$  by an appropriate smaller subset  $B_j$ .

First, we define the inner part as

$$(27) \quad \hat{B}_j^{\text{inn}} = \left\{ \nu \in P_{\mathbf{L}_{\text{inn}}}^+ \mid \begin{array}{l} (1) \quad \rho_{\mathbf{H}_j} \pm 2i_*(w\nu) \in \mathbf{R}_j \text{ for all } w \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \\ (2) \quad j\xi + i_*\nu \in P_{\mathbf{L}} \end{array} \right\}.$$

After that we define the outer part as

$$(28) \quad \hat{B}_j^{\text{out}} = \left\{ \mu \in \text{Ker } h_j^* \cap P_{\mathbf{G}}^+ \mid \begin{array}{l} \rho_{\mathbf{H}_j} - h_j^*(w_{\mathbf{L}_{\text{out}}}\mu) - i_*(w_{\mathbf{L}_{\text{inn}}}\nu) + i_*(w'_{\mathbf{L}_{\text{inn}}}\nu') \in \mathbf{R}_j \\ \text{for all } \nu, \nu' \in \hat{B}_j^{\text{inn}}, w_{\mathbf{L}_{\text{out}}} \in \mathbf{W}_{\mathbf{L}_{\text{out}}}, \text{ and } w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \end{array} \right\}.$$

And finally, we consider the set

$$(29) \quad \hat{B}_j = \hat{B}_j^{\text{out}} + j\xi + i_*(\hat{B}_j^{\text{inn}}).$$

*Remark 5.5.* In both definitions (27) and (28) we can replace all terms of the form  $i_*(w\nu)$  with  $w \in \mathbf{W}_{\mathbf{L}_{\text{inn}}}$  by  $wi_*(\nu)$  and allow  $w$  to run through the entire group  $\mathbf{W}_{\mathbf{L}}$ . Indeed, this follows from the decomposition  $\mathbf{W}_{\mathbf{L}} = \mathbf{W}_{\mathbf{L}_{\text{out}}} \times \mathbf{W}_{\mathbf{L}_{\text{inn}}}$  together with the fact that  $\mathbf{W}_{\mathbf{L}_{\text{inn}}}$  acts trivially on the image of  $i_*$ .

**5.5. Very special representatives.** In this section we will define a certain class of elements of the set  $\text{SR}_{\mathbf{H}}^{\mathbf{M}}$  and using them define a subblock  $B_j \subset \hat{B}_j$ . In fact,  $B_j = \hat{B}_j$  unless  $\mathbf{G}$  is of type  $A$ .

Recall that  $\mathbf{L}_{\text{out}} = \text{SL}_k$ , see (24). We will use the following representation of the weight lattice of  $\text{SL}_k$ :

$$P_{\text{SL}_k} = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{Q}^k \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z} \text{ for all } 1 \leq i \leq k-1 \text{ and } \sum_{i=1}^k \lambda_i = 0\},$$

where the simple roots and the fundamental weights are given by

$$\alpha_t = \left( \underbrace{0, \dots, 0}_{t-1}, 1, -1, \underbrace{0, \dots, 0}_{k-t-1} \right), \quad \omega_t = \left( \underbrace{\frac{k-t}{k}, \dots, \frac{k-t}{k}}_t, \underbrace{-\frac{t}{k}, \dots, -\frac{t}{k}}_{k-t} \right).$$

*Remark 5.6.* Note that this representation fixes the scaling of the scalar product as  $\alpha_t^2 = 2$  for all  $1 \leq t \leq k-1$ . From now on we fix this scaling.

Let  $\mathbf{H} = \mathbf{H}_a$  for some  $a$ ,  $1 \leq a \leq b$ . For each  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$  define

$$(30) \quad \phi(v) := \frac{(\xi, \rho - v\rho)}{k(\xi, \omega_1)} \left( 1 - k \frac{(\xi, \omega_1)^2}{\xi^2} \right).$$

**Definition 5.7.** An element  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$  is **very special** if  $\phi(v)$  is a positive integer.

**Lemma 5.8.** *If  $\mathbf{G}$  is a group of type  $B$ ,  $C$  or  $D$ , then there are no very special elements.*

*Proof.* Consider the standard numbering of vertices. Let  $\beta = \alpha_k$ . Note that if we take  $D_{\text{out}}$  to be empty then we have nothing to check (since we assumed  $a \geq 1$ ). This means that we only have to consider the case when  $D_{\text{out}}$  consists of vertices from 1 to  $k-1$ .

First, assume that either  $k \leq n-1$  for type  $B$  and  $k \leq n-2$  for type  $D$  or any  $k$  for type  $C$ . Then  $(\xi, \omega_1) = 1$ ,  $\xi^2 = k$  and we see that the second factor in (30) vanishes, hence  $\phi(v) = 0$ . In the remaining cases ( $k = n$  for type  $B$  and  $k = n$  for type  $D$ ) we have  $(\xi, \omega_1) = 1/2$ ,  $\xi^2 = n/4$ , and  $k = n$ , so the second factor vanishes as well.  $\square$

*Remark 5.9.* It seems plausible that for types  $E$ ,  $F$  and  $G$  there are no very special elements as well, although we have not checked this. On the contrary, for type  $A$

$$\phi(v) = (\xi, \rho - v\rho)/(n+1-k),$$

so very special elements correspond to  $v \in S_{n+1}$  such that  $v(n+1) = k$ .

Now we are ready to define the block — we just set

$$(31) \quad \begin{aligned} \mathbf{B}_j^{\text{out}} &= \{\lambda \in \hat{\mathbf{B}}_j^{\text{out}} \mid (\lambda + v\rho - \rho, \alpha_1 + \dots + \alpha_{k-1}) < \phi(v) \text{ for all very special } v\}, \\ \mathbf{B}_j^{\text{inn}} &= \hat{\mathbf{B}}_j^{\text{inn}}, \\ \mathbf{B}_j &= \mathbf{B}_j^{\text{out}} + j\xi + i_*(\mathbf{B}_j^{\text{inn}}), \end{aligned}$$

Further we will show that the block  $\mathbf{B}_j$  defined by (31) is exceptional if the outer part of it, considered as a set of Young diagrams, is closed under passing to a subdiagram. In fact, we will prove part (a) of the criterion 3.13 in section 6 (without additional conditions). And part (b) of this criterion will be proved in section 7 provided the above condition holds. Finally, the above condition will be verified for groups of type  $BCD$  by direct computation in section 9.

**5.6. Exceptional collections.** Before we proceed to the proof that the constructed blocks are exceptional we will explain how one can make these blocks smaller in order to achieve semiorthogonality of the subcategories of  $\mathcal{D}^b(X)$  generated by the corresponding equivariant bundles.

First, we define subsets  $\bar{\mathbf{B}}_j^{\text{inn}} \subset \mathbf{B}_j^{\text{inn}}$  by the formula

$$(32) \quad \bar{\mathbf{B}}_j^{\text{inn}} = \left\{ \nu \in \mathbf{B}_j^{\text{inn}} \mid \begin{array}{l} \text{for all } j' < j, \nu' \in \bar{\mathbf{B}}_{j'}^{\text{inn}}, \text{ and } w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}} \\ \text{one has } \rho_{\mathbf{H}_{j'}} - (j - j')\xi - w_{\mathbf{L}_{\text{inn}}} i_* \nu + w'_{\mathbf{L}_{\text{inn}}} i_* \nu' \in \mathbf{R}_{j'}^* \end{array} \right\}.$$

Note that the above formula is recursive — it describes  $\bar{\mathbf{B}}_j^{\text{inn}}$  in terms of all  $\bar{\mathbf{B}}_{j'}^{\text{inn}}$  with  $j' < j$ . We also set

$$(33) \quad \bar{\mathbf{B}}_j^{\text{out}} = \left\{ \lambda_0 \in \mathbf{B}_j^{\text{out}} \mid \begin{array}{l} \text{for all } j' < j, \nu \in \bar{\mathbf{B}}_j^{\text{inn}}, \nu' \in \bar{\mathbf{B}}_{j'}^{\text{inn}}, w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}}, \text{ and } w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}} \\ \text{one has } \rho_{\mathbf{H}_{j'}} - h_{j'}^*(w_{\mathbf{L}} \lambda_0 + (j - j')\xi) - w_{\mathbf{L}_{\text{inn}}} i_* \nu + w'_{\mathbf{L}_{\text{inn}}} i_* \nu' \in \mathbf{R}_{j'}^* \end{array} \right\}$$

Note that by Remark 5.5, we can let the elements  $w_{\mathbf{L}_{\text{inn}}}$  and  $w'_{\mathbf{L}_{\text{inn}}}$  run through the entire group  $\mathbf{W}_{\mathbf{L}}$  in the definitions (32) and (33). Finally, we set

$$(34) \quad \bar{\mathbf{B}}_j = \bar{\mathbf{B}}_j^{\text{out}} + j\xi + i_* \bar{\mathbf{B}}_j^{\text{inn}},$$

and define the subcategory

$$\mathcal{A}_j := \langle \mathcal{U}^\lambda \rangle_{\lambda \in \bar{\mathbf{B}}_j}.$$

**Theorem 5.10.** *The collection of subcategories  $\{\mathcal{A}_j\}_{j \in \mathbf{J}}$  ordered by increasing of  $j$  is semiorthogonal.*

*Proof.* Assume that  $j' < j$ . Let  $\lambda_0 \in \bar{\mathbf{B}}_j^{\text{out}}$ ,  $\lambda'_0 \in \bar{\mathbf{B}}_{j'}^{\text{out}}$ ,  $\nu \in \bar{\mathbf{B}}_j^{\text{inn}}$ ,  $\nu' \in \bar{\mathbf{B}}_{j'}^{\text{inn}}$ . We have to check that

$$\text{Ext}^\bullet(\mathcal{U}^{\lambda_0 + j\xi + i_* \nu}, \mathcal{U}^{\lambda'_0 + j'\xi + i_* \nu'}) = 0.$$

By Corollary 2.16 we have to check that for any  $\mathbf{L}$ -dominant weight

$$\mu \in \text{Conv}(\lambda'_0 - w_{\mathbf{L}} \lambda_0 + (j' - j)\xi + i_* \nu' - w_{\mathbf{L}} i_* \nu)_{w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}}$$

the sum  $\mu + \rho_{\mathbf{G}}$  is singular. Note that  $h_{j'}^*(\lambda'_0) = 0$  since  $\lambda'_0 \in \bar{\mathbf{B}}_{j'}^{\text{out}} \subset \text{Ker } h_{j'}^*$ , hence

$$h_{j'}^*(\mu + \lambda'_0 - w_{\mathbf{L}} \lambda_0 + (j' - j)\xi + i_* \nu' - w_{\mathbf{L}} i_* \nu) = \rho_{\mathbf{H}_{j'}} - h_{j'}^*(w_{\mathbf{L}} \lambda_0 - (j - j')\xi) + i_* \nu' - w_{\mathbf{L}} i_* \nu.$$

By definition of  $\bar{\mathbf{B}}_j^{\text{out}}$ , all these weights for  $w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}$  lie in the interior of the core  $\mathbf{R}_{j'}$ , hence  $h_{j'}^*(\mu + \rho) \in \mathbf{R}_{j'}^*$ , and so by Lemma 5.3  $h_{j'}^*(\mu + \rho)$  is singular. But the map  $h_{j'}^*$  preserves regularity, hence  $\mu + \rho$  is singular as well.  $\square$

## 6. VERIFICATION OF THE INVARIANCE CONDITION

In this section we prove that the blocks  $B_j$  and  $\bar{B}_j$  constructed in section 5 satisfy the invariance condition (part (a) of the criterion 3.13).

First, we will need the following simple fact. Assume that  $\mathbf{H} \subset \mathbf{H}'$  is a pair of semisimple groups corresponding to subdiagrams  $D_{\mathbf{H}} \subset D_{\mathbf{H}'}$  of the Dynkin diagrams such that  $D_{\mathbf{H}'} \setminus D_{\mathbf{H}}$  consists only of one vertex. Let  $\alpha$  be the corresponding simple root and  $\eta$  the corresponding fundamental weight of  $\mathbf{H}'$ .

**Lemma 6.1.** *There is a positive integer  $k = k_{\mathbf{H}', \mathbf{H}}$  such that*

$$\rho_{\mathbf{H}'} - k\eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}}.$$

Moreover, for all  $0 < c < k$  the weight  $\rho_{\mathbf{H}'} - c\eta$  is singular.

*Proof.* Let us denote the embedding  $\mathbf{H} \rightarrow \mathbf{H}'$  by  $h$ . Then as we know  $h^* \rho_{\mathbf{H}'} = \rho_{\mathbf{H}}$  and  $\text{Ker } h^* = \mathbb{Z}\eta$ . Therefore,

$$h^* w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} = -h^* w_0^{\mathbf{H}} \rho_{\mathbf{H}'} = -w_0^{\mathbf{H}} \rho_{\mathbf{H}} = \rho_{\mathbf{H}},$$

so

$$\rho_{\mathbf{H}'} - w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} = k\eta$$

for some  $k \in \mathbb{Z}$ . Moreover, the LHS is a sum of positive roots by Lemma 2.3, hence  $(k\eta, \eta) > 0$ , hence  $k$  is positive. This proves the first statement.

For the second let us check first that any weight  $\rho_{\mathbf{H}'} - c\eta$  with  $0 < c < k$  lies in the convex hull of  $\rho_{\mathbf{H}'} - \eta$  and  $w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} + \eta$ , hence in the interior of any core in  $P_{\mathbf{H}'} \otimes \mathbb{R}$ . Indeed,  $\rho_{\mathbf{H}'} - \eta$  is dominant, so taking any strictly dominant  $\delta$  we see that  $(\delta, \rho_{\mathbf{H}'} - \eta) = (\delta, \rho_{\mathbf{H}'}) - (\delta, \eta) < (\delta, \rho_{\mathbf{H}'})$ , hence  $\rho_{\mathbf{H}'} - \eta \in \mathbf{R}_{\delta}^*$  by Lemma 5.2. On the other hand,

$$\rho_{\mathbf{H}'} - (k-1)\eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} + \eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} (\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} w_0^{\mathbf{H}} \eta) = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} (\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} \eta).$$

Since  $-w_0^{\mathbf{H}'} \eta$  is a fundamental weight of  $\mathbf{H}'$ , the same argument as above shows that

$$\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} \eta = \rho_{\mathbf{H}'} - (-w_0^{\mathbf{H}'} \eta) \in \mathbf{R}_{\delta}^*.$$

Hence, we obtain that  $\rho_{\mathbf{H}'} - (k-1)\eta$  is also in  $\mathbf{R}_{\delta}^*$ . Since  $\mathbf{R}_{\delta}^*$  is convex, we see that  $\rho_{\mathbf{H}'} - c\eta$  is in the interior of the core for all  $1 \leq c \leq k-1$ . Hence, all these weights are singular by Lemma 5.3.  $\square$

*Remark 6.2.* One can also deduce the claim geometrically. Consider the Grassmannian of  $\mathbf{H}'$  corresponding to the root  $\beta$ . Then its Picard group is  $\mathbb{Z}$  and the pullback of its generator to the flag variety of  $\mathbf{H}'$  is the line bundle corresponding to the weight  $\eta$ . It is known that the canonical class of the Grassmannian is given by the weight  $w_{\mathbf{H}}^0 w_{\mathbf{H}'}^0 \rho - \rho$ . On the other hand, it is equal to the line bundle corresponding to the weight  $-k\eta$  for some  $k \in \mathbb{Z}$ . This gives the equality. Having all this, the singularity of weights  $\rho - c\eta$  with  $0 < c < k$  is clear. Indeed by the Borel–Bott–Weil the singularity of  $\rho - c\eta$  is equivalent to the vanishing of the cohomology of  $-c\eta$ , which indeed vanishes by Kodaira vanishing theorem.

Now we can verify the invariance condition.

**Proposition 6.3.** *Let  $\kappa \in \text{OP}_1(B_j)$ ,  $v \in \text{OP}_2(B_j)$ . Then  $\kappa \in \text{Ker } h_j^*$  and  $v \in \mathbf{W}_{\mathbf{H}_j}$ . In particular,  $v\kappa = \kappa$ .*

*Proof.* Take arbitrary  $\lambda, \lambda' \in B_j$ . Then  $\lambda = \lambda_0 + p\xi + i_*\nu$ ,  $\lambda' = \lambda'_0 + p\xi + i_*\nu'$ , with  $\lambda_0, \lambda'_0 \in B_j^{\text{out}}$  and  $\nu, \nu' \in B_j^{\text{inn}}$ . Note that for any  $w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}$  we have

$$(35) \quad h_j^*(\rho + \lambda' - w_{\mathbf{L}}\lambda) = h_j^*(\rho + \lambda'_0 + i_*\nu' - w_{\mathbf{L}}\lambda_0 - w_{\mathbf{L}}i_*\nu) = h_j^*(\rho - w_{\mathbf{L}}\lambda_0) + i_*\nu' - w_{\mathbf{L}}i_*\nu$$

since  $\lambda'_0 \in \text{Ker } i^*$  and  $h_j \circ i = i$ . So, by definition of  $B_j$  (using Remark 5.5) we conclude that the weight (35) is in  $\mathbf{R}_{\delta}$ .

Let  $(\kappa, v) \in \text{OP}(\text{B}_j)$ , that is  $(\kappa, v) \in \text{OP}(\lambda, \lambda')$  for some  $\lambda, \lambda' \in \text{B}_j$ . By definition of the output set the weight

$$\mu := v(\kappa + \rho) - \rho \in \text{Conv}(\lambda' - w_{\mathbf{L}}\lambda)_{w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}}$$

is  $\mathbf{L}$ -dominant and  $\mu + \rho$  is  $\mathbf{G}$ -regular. Moreover,  $h_j^*(\mu + \rho)$  is in the convex hull of the weights (35) (where  $w_{\mathbf{L}}$  runs through  $\mathbf{W}_{\mathbf{L}}$ ) hence is in the core  $\mathbf{R}_{\delta}$ . So, Proposition 6.4 below applies and we conclude that  $\kappa \in \text{Ker } h_j^*$ ,  $v \in \mathbf{W}_{\mathbf{H}_j}$ .  $\square$

**Proposition 6.4.** *Assume that a weight  $\mu \in P_{\mathbf{L}}$  satisfies*

$$(36) \quad \mu \in P_{\mathbf{L}}^+, \quad \mu + \rho \in P_{\mathbf{G}}^{\text{reg}}, \quad h_a^*(\mu + \rho) \in \mathbf{R}_{\delta},$$

for some  $a$ ,  $0 \leq a \leq b$ . Let also  $\mu = v(\kappa + \rho) - \rho$  be the unique presentation of  $\mu$  with  $\kappa \in P_{\mathbf{G}}^+$  and  $v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$ . Then

$$v \in \text{SR}_{\mathbf{G}}^{\mathbf{L}} \cap \mathbf{W}_{\mathbf{H}_a} \quad \text{and} \quad \kappa \in P_{\mathbf{G}}^+ \cap \text{Ker } h_a^*.$$

In particular,  $v\kappa = \kappa$ .

*Proof.* To simplify the notation we write  $\mathbf{H}$  instead of  $\mathbf{H}_a$  and  $h$  instead of  $h_a$ . Set  $\mathbf{M} = \mathbf{L} \cap \mathbf{H}$ . Note that  $h^*$  takes regular  $\mathbf{L}$ -dominant weights of  $P_{\mathbf{G}}$  to regular  $\mathbf{M}$ -dominant weights of  $P_{\mathbf{H}}$ , hence  $h^*(\mu + \rho)$  is regular and  $\mathbf{M}$ -dominant. On the other hand,  $h^*(\mu + \rho) \in \mathbf{R}_{\delta}$ , so Lemma 5.3 implies that  $h^*(\mu + \rho) = v\rho_{\mathbf{H}}$  with  $v \in \mathbf{W}_{\mathbf{H}}$ . Thus,  $v\rho_{\mathbf{H}}$  is  $\mathbf{M}$ -dominant, so we have  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$ . Further,  $v\rho_{\mathbf{H}} = h^*(v\rho)$ , hence  $h^*(\mu + \rho - v\rho) = 0$ . Denoting

$$\kappa = \mu + \rho - v\rho$$

we see that  $\kappa \in \text{Ker } h^*$  and  $\mu = v\rho - \rho + \kappa$ . Since  $\kappa \in \text{Ker } h^*$  and  $v \in \mathbf{W}_{\mathbf{H}}$  we have  $v\kappa = \kappa$ , so  $\mu$  can be written as  $v(\kappa + \rho) - \rho$ . So it remains to check that  $\kappa$  is  $\mathbf{G}$ -dominant.

To check the dominance of a weight we should check that its inner products with all simple roots are nonnegative. We divide the simple roots into three groups and consider them one-by-one.

**Case 1:** the simple roots of  $\mathbf{H}$ . If  $\alpha \in D_{\mathbf{H}}$  then  $(\kappa, \alpha) = 0$  since  $\kappa \in \text{Ker } h^*$ .

**Case 2:** the simple roots of  $\mathbf{G}$  not adjacent to  $D_{\mathbf{H}}$ . If  $\alpha$  is such a root then  $v^{-1}\alpha = \alpha$  since  $v \in \mathbf{W}_{\mathbf{H}}$ , hence  $(v\rho, \alpha) = (\rho, v^{-1}\alpha) = (\rho, \alpha)$ , therefore  $(\kappa, \alpha) = (\mu, \alpha) \geq 0$ . The last inequality follows from  $\mathbf{L}$ -dominance of  $\mu$  since simple roots not adjacent to  $D_{\mathbf{H}}$  are roots of  $\mathbf{L}$ .

**Case 3:** the simple root adjacent to  $D_{\mathbf{H}}$ . Let  $\alpha$  be such a root and let  $\mathbf{H}'$  be the reductive subgroup of  $\mathbf{G}$  such that  $D_{\mathbf{H}'} = D_{\mathbf{H}} \cup \{\alpha\}$ . Let  $\eta \in P_{\mathbf{H}'}$  be the fundamental weight of  $\mathbf{H}'$  corresponding to the root  $\alpha$ . Let  $h' : \mathbf{H}' \rightarrow \mathbf{G}$  be the embedding, and let  $h$  denote the embeddings  $\mathbf{H} \rightarrow \mathbf{H}'$  and  $\mathbf{H} \rightarrow \mathbf{G}$ . Note that  $\text{Ker}(h^* : P_{\mathbf{H}'} \rightarrow P_{\mathbf{H}}) = \mathbb{Z}\eta$ .

Note that  $h^*(h')^*(\mu + \rho) = h^*(\mu + \rho) = v\rho_{\mathbf{H}} = h^*(v\rho_{\mathbf{H}'})$ , hence  $(h')^*(\mu + \rho) = v\rho_{\mathbf{H}'} + c\eta = v(\rho_{\mathbf{H}'} + c\eta)$ . It is enough to show that  $c \geq 0$ . Indeed, since  $\alpha$  is a root of  $\mathbf{H}'$  we have  $\alpha = h'_*\alpha$ , so

$$(\kappa, \alpha) = (\kappa, h'_*\alpha) = ((h')^*\kappa, \alpha) = ((h')^*(\mu + \rho - v\rho), \alpha) = (v\rho_{\mathbf{H}'} + c\eta - v\rho_{\mathbf{H}'}, \alpha) = c(\eta, \alpha) = c\alpha^2/2 \geq 0$$

and we are done. So, assume that  $c < 0$ . Since  $v^{-1}(h')^*(\mu + \rho)$  is regular, Lemma 6.1 implies that  $v^{-1}(h')^*(\mu + \rho) = \rho_{\mathbf{H}'} + c\eta = -w_0^{\mathbf{H}}\rho_{\mathbf{H}'} - c'\eta$  with  $c' \geq 0$ . Then

$$(h')^*\mu = v(-w_0^{\mathbf{H}}\rho_{\mathbf{H}'} - c'\eta) - (h')^*\rho = -vw_0^{\mathbf{H}}\rho_{\mathbf{H}'} - \rho_{\mathbf{H}'} - c'\eta.$$

Let us check that the scalar product of this weight with  $\alpha$  is always negative. Indeed,  $(\rho_{\mathbf{H}'}, \alpha) > 0$  since  $\rho_{\mathbf{H}'}$  is a strictly dominant weight of  $\mathbf{H}'$ . Further, the root  $w_0^{\mathbf{H}}v^{-1}\alpha$  is positive since  $(\eta, w_0^{\mathbf{H}}v^{-1}\alpha) = (vw_0^{\mathbf{H}}\eta, \alpha) = (\eta, \alpha) > 0$ . Therefore,  $(vw_0^{\mathbf{H}}\rho_{\mathbf{H}'}, \alpha) = (\rho_{\mathbf{H}'}, w_0^{\mathbf{H}}v^{-1}\alpha) > 0$ . Finally,  $(c'\eta, \alpha) \geq 0$  since  $c' \geq 0$ . Thus, we see that

$$((h')^*\mu, \alpha) < 0.$$

But this is equal to  $(\mu, \alpha)$  which is nonnegative since  $\mu$  is  $\mathbf{L}$ -dominant. This contradiction shows that we actually have  $c \geq 0$  which completes the proof.  $\square$

## 7. ADAPTED WEIGHTS AND COMPATIBILITY CONDITION

Let  $L$  be a reductive algebraic group. For any subset  $S \subset P_L^+$  of the set of dominant weights of  $L$  we denote by  $\text{Rep}_S(L)$  the subcategory of  $\text{Rep}(L)$  consisting of direct sum of irreducible representations with highest weights in  $S$ . We also denote by  $\Pi_S : \text{Rep}(L) \rightarrow \text{Rep}(L)$  the corresponding projector (that leaves only representations in  $\text{Rep}_S(L)$ ).

A morphism  $f : V_1 \rightarrow V_2$  in  $\text{Rep}(L)$  is called an  $S$ -isomorphism if  $\Pi_S(f) : \Pi_S(V_1) \rightarrow \Pi_S(V_2)$  is an isomorphism. In other words,  $f$  is an  $S$ -isomorphism if it induces an isomorphism on  $\lambda$ -isotypical components for any  $\lambda \in S$ .

We say that a pair of  $L$ -dominant weights  $(\kappa, \lambda)$  is **adapted to  $S$**  (or  $S$ -adapted) if the natural map

$$(37) \quad V_L^{\kappa+\lambda} \otimes V_L^\mu \rightarrow V_L^\kappa \otimes V_L^\lambda \otimes V_L^\mu \rightarrow V_L^\kappa \otimes \Pi_S(V_L^\lambda \otimes V_L^\mu)$$

is an  $S$ -isomorphism for any  $\mu \in S$ .

The goal of this section is to show that for all  $(\kappa, v) \in \text{OP}_1(\mathbf{B}) \times \text{OP}_2(\mathbf{B})$  the pair  $(\kappa, v\rho - \rho)$  (considered as weights of the Levi group  $\mathbf{L}$ ) is  $\mathbf{B}$ -adapted for either  $\mathbf{B} = \mathbf{B}_j$  or  $\mathbf{B} = \bar{\mathbf{B}}_j$ . In fact, we will prove the following more general statement.

Take any  $a$  with  $0 \leq a \leq b$ , any  $j \in \mathbb{Q}$  and any subsets  $\mathbf{B}^{\text{inn}} \subset P_{\mathbf{L}_{\text{inn}}}^+$ ,  $\mathbf{B}^{\text{out}} \subset P_{\mathbf{G}}^+ \cap \text{Ker } h^*$  (we will write  $\mathbf{H}$  for  $\mathbf{H}_a$  and  $h$  for  $h_a$  for brevity) such that  $j\xi + i_*\mathbf{B}^{\text{inn}} \subset P_{\mathbf{L}}$ . Set

$$\mathbf{B} = \mathbf{B}^{\text{out}} + j\xi + i_*\mathbf{B}^{\text{inn}}.$$

Let us denote  $\mathbf{M} = \mathbf{M} \cap \mathbf{H}$ . Note that the elements of  $\mathbf{B}^{\text{out}}$  can be viewed as Young diagrams.

**Theorem 7.1.** *Assume that the set  $\mathbf{B}^{\text{out}}$  has the following two properties:*

- (1) *for all  $\lambda \in \mathbf{B}^{\text{out}}$  and all very special  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$  we have  $(\lambda + v\rho - \rho, \alpha_1 + \dots + \alpha_{k-1}) < \phi(v)$ ;*
- (2) *the set  $\mathbf{B}^{\text{out}}$  is closed under passing to Young subdiagrams.*

*Then for any  $\kappa \in P_{\mathbf{G}}^+ \cap \text{Ker } h^*$  and any  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$  the pair  $(\kappa, v\rho - \rho)$  is  $\mathbf{B}$ -adapted.*

**Corollary 7.2.** *Assume for some  $j \in \mathbf{J}$  the set  $\mathbf{B}_j^{\text{out}}$  (resp.,  $\bar{\mathbf{B}}_j^{\text{out}}$ ) is closed under passing to Young subdiagrams. Then the block  $\mathbf{B}_j$  (resp.,  $\bar{\mathbf{B}}_j$ ) is exceptional.*

*Proof.* Set  $\mathbf{B} = \mathbf{B}_j$  (resp.,  $\bar{\mathbf{B}}_j$ ). It is enough to check the two conditions of Proposition 3.13 for  $\mathbf{B}$ . The invariance condition holds for this block by Proposition 6.3. To check the compatibility condition we can apply Theorem 7.1. The first condition of this theorem holds by the definition (31) of the block  $\mathbf{B}_j$ , while the second holds by assumption. It remains to observe that for any pair  $\kappa \in \text{OP}_1(\mathbf{B})$ ,  $v \in \text{OP}_2(\mathbf{B})$  we have  $\kappa \in P_{\mathbf{G}}^+ \cap \text{Ker } h^*$  and  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$ . Hence, Theorem 7.1, applied to  $\mathbf{B}$  and a pair  $\kappa \in \text{OP}_1(\mathbf{B})$ ,  $v \in \text{OP}_2(\mathbf{B})$ , implies that the compatibility condition is satisfied for  $\mathbf{B}$ .  $\square$

Unfortunately, we were not able to find an abstract way of checking that  $\mathbf{B}_j$  or  $\bar{\mathbf{B}}_j^{\text{out}}$  is closed under passing to Young subdiagrams. So, we will check it for classical groups in section 9 as a result of an explicit description of the blocks.

**7.1. Preparations.** We start with a description of the connected component of the center of  $\mathbf{L}$ .

**Lemma 7.3.** *Let  $\mathbf{Z} \subset \mathbf{L}$  be the connected component of the center of  $\mathbf{L}$ . Then  $\mathbf{Z} \cong \mathbb{G}_m$  and the map  $P_{\mathbf{L}} \rightarrow P_{\mathbf{Z}} = \mathbb{Z}$ , induced by the embedding  $\mathbf{Z} \rightarrow \mathbf{L}$ , is given by the scalar product with the minimal rational multiple  $c\xi$  of  $\xi$ , such that  $(c\xi, -)$  is an integral valued function on  $P_{\mathbf{L}}$ .*

*Proof.* First, note that  $\mathbf{Z} \cong \mathbb{G}_m$  since it is a 1-dimensional (since  $\mathbf{P}$  is maximal) connected commutative reductive group. As a consequence,  $P_{\mathbf{Z}} \cong \mathbb{Z}$ . Since the map  $P_{\mathbf{L}} \rightarrow P_{\mathbf{Z}}$  is dual to the embedding of  $\mathbf{Z}$  into a maximal torus of  $\mathbf{L}$ , it is surjective. Note also that the adjoint representation of the semisimple part of  $\mathbf{L}$  is a trivial representation of  $\mathbf{Z}$ , hence all simple roots of  $\mathbf{L}$  are mapped to zero. This implies that the

map is given by the scalar product with a multiple  $c\xi$  of  $\xi$ . Moreover, since the scalar product should be a map to  $\mathbb{Z}$ , it follows that  $(c\xi, -)$  should be an integral function on  $P_{\mathbf{L}}$  (and in particular,  $c$  should be rational since the scalar product has rational values on the weight lattice), and the surjectivity of the map implies that  $c$  is minimal with this property.  $\square$

Now take any  $\kappa \in \text{Ker } h^* \cap P_{\mathbf{G}}^+$ ,  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$  and  $\mu \in \mathbf{B}$ , and consider the morphisms

$$(38) \quad V_{\mathbf{L}}^{\kappa+v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu} \rightarrow V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu} \rightarrow V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathbf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu}).$$

To get hold of these tensor products we consider the diagram of groups

$$\begin{array}{ccc} & \mathbf{L}_{\text{out}} \times \mathbb{G}_m \times \mathbf{L}_{\text{inn}} & \\ \varpi \swarrow & & \searrow \pi \\ \text{GL}_k \times \mathbb{G}_m \times \mathbf{L}_{\text{inn}} & & \mathbf{L} \end{array}$$

where  $\pi$  and  $\varpi$  are defined as follows. Let  $\mathbf{Z}$  be the connected component of the center of  $\mathbf{L}$ . We use an isomorphism  $\mathbf{Z} \simeq \mathbb{G}_m$  as in Lemma 7.3, so that the corresponding projection  $P_{\mathbf{L}} \rightarrow P_{\mathbf{Z}} = \mathbb{Z}$  is given by  $\lambda \mapsto (c\xi, \lambda)$ , where  $c$  is the minimal positive rational number such that  $(c\xi, \lambda) \in \mathbb{Z}$  for all  $\lambda \in P_{\mathbf{L}}$ . The morphism  $\pi$  is induced by the embeddings  $o : \mathbf{L}_{\text{out}} \rightarrow \mathbf{L}$ ,  $i : \mathbf{L}_{\text{inn}} \rightarrow \mathbf{L}$  and by the isomorphism  $\mathbb{G}_m \cong \mathbf{Z}$ . The restriction of  $\varpi$  to  $\mathbf{L}_{\text{out}} = \text{SL}_k$  (resp.,  $\mathbf{L}_{\text{inn}}$ ) is given by the natural embedding  $\text{SL}_k \subset \text{GL}_k$  (resp., the identity map to  $\mathbf{L}_{\text{inn}}$ ). Finally, the restriction of  $\varpi$  to  $\mathbb{G}_m$  is given by  $z \mapsto (z^{(c\xi, \omega_1)} \times 1, z, 1)$ . We will express the pullbacks by  $\pi$  of all representations involved in (38) as pullbacks by  $\varpi$  of representations of  $\text{GL}_k \times \mathbb{G}_m \times \mathbf{L}_{\text{inn}}$ , and then will reduce the claim to an analogous statement about representations of  $\text{GL}_k$ . Recall that irreducible representations of  $\text{GL}_k$  are numbered by nonincreasing sequences of integers of length  $k$ .

**Lemma 7.4.** *We have*

$$\pi^* V_{\mathbf{L}}^{\lambda} = V_{\mathbf{L}_{\text{out}}}^{o^* \lambda} \otimes V_{\mathbb{G}_m}^{(c\xi, \lambda)} \otimes V_{\mathbf{L}_{\text{inn}}}^{i^* \lambda}.$$

On the other hand,

$$\varpi^*(V_{\text{GL}_k}^{\kappa_{\bullet}} \otimes V_{\mathbb{G}_m}^z \otimes V_{\mathbf{L}_{\text{inn}}}^{\nu}) = V_{\mathbf{L}_{\text{out}}}^{\kappa} \otimes V_{\mathbb{G}_m}^{(c\xi, \omega_1) \sum_{i=1}^k \kappa_i + z} \otimes V_{\mathbf{L}_{\text{inn}}}^{\nu}.$$

*Proof.* This is straightforward (to compute the central character of  $\pi^* V_{\mathbf{L}}^{\lambda}$  use Lemma 7.3).  $\square$

Now we will need more precise information about representations entering into (38).

**Lemma 7.5.** *Let  $\kappa \in \text{Ker } h^* \cap P_{\mathbf{G}}^+$ . Then there exists a unique nonincreasing sequence of integers  $\kappa_{\bullet} = (\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_a \geq \kappa_{a+1} = \dots = \kappa_k = 0)$  such that*

$$\pi^* V_{\mathbf{L}}^{\kappa} = \varpi^* V_{\text{GL}_k}^{\kappa_{\bullet}}.$$

*Proof.* By definition,  $\kappa$  is a nonnegative linear combination of  $\omega_1, \dots, \omega_a$ . Let  $\kappa_1 - \kappa_2, \kappa_2 - \kappa_3, \dots, \kappa_{a-1} - \kappa_a$  and  $\kappa_a$  be the coefficients. Then  $\kappa_1 \geq \dots \geq \kappa_a \geq 0$ . Extending this sequence by  $\kappa_{a+1} = \dots = \kappa_k = 0$  we obtain a sequence  $\kappa_{\bullet}$ . To prove the required isomorphism we use Lemma 7.4. By this Lemma, we only have to check that  $(c\xi, \kappa) = (c\xi, \omega_1) \sum_{i=1}^k \kappa_i$ . For this we note that for  $i < b$  we have  $\alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}$ , hence  $(c\xi, \omega_i) = i(c\xi, \omega_1)$ , so

$$(c\xi, \kappa) = (c\xi, \omega_1) \sum_{i=1}^a i(\kappa_i - \kappa_{i-1}) = (c\xi, \omega_1) \sum_{i=1}^a \kappa_i = (c\xi, \omega_1) \sum_{i=1}^k \kappa_i,$$

as required.  $\square$

**Lemma 7.6.** *Let  $v \in \text{SR}_{\mathbf{H}}^{\mathbf{M}}$ . Set  $\nu_v = i^*(v\rho - \rho)$ . Then there exists a unique sequence of integers  $0 = \tau_1 = \dots = \tau_a \geq \tau_{a+1} \geq \dots \geq \tau_k$  such that*

$$\pi^* V_{\mathbf{L}}^{v\rho-\rho} = \varpi^*(V_{\text{GL}_k}^\tau \otimes V_{\mathbb{G}_m}^{z(v)} \otimes V_{\mathbf{L}_{\text{inn}}}^{\nu_v}),$$

where

$$(39) \quad z(v) = (v\rho - \rho, c\xi)(1 - k(\omega_1, \xi)^2/\xi^2).$$

*Proof.* Consider the restriction  $o^*(v\rho - \rho)$ . It is a weight of  $\text{SL}_k$ . A weight of  $\text{SL}_k$  can be thought of as a weight of  $\text{GL}_k$  up to adding a central character. In other words, it is given by a nonincreasing sequence of integers up to a simultaneous translation. Consider the sequence  $\tau_1 \geq \dots \geq \tau_k$  representing  $o^*(v\rho - \rho)$  such that  $\tau_1 = 0$ . Note that  $v\rho - \rho$  is orthogonal to  $\alpha_1, \dots, \alpha_{a-1}$  (because these roots are orthogonal to the roots of  $\mathbf{H}$  and hence are  $v$ -invariant), hence  $\tau_1 = \tau_2 = \dots = \tau_a$ .

Further, we denote by  $\nu_v$  the weight  $i^*(v\rho - \rho)$ . Then the representations  $\pi^* V_{\mathbf{L}}^{v\rho-\rho}$  and  $\varpi^*(V_{\text{GL}_k}^\tau \otimes V_{\mathbf{L}_{\text{inn}}}^{\nu_v})$  have the same restrictions to  $\mathbf{L}_{\text{out}}$  and  $\mathbf{L}_{\text{inn}}$ , so it remains to compare the central characters. First, the central character of  $V_{\mathbf{L}}^{v\rho-\rho}$  is  $(c\xi, v\rho - \rho)$ . Further, the central character of  $V_{\text{GL}_k}^\tau$  is  $(c\xi, \omega_1) \sum \tau_i$ , while the central character of  $V_{\mathbf{L}_{\text{inn}}}^{\nu_v}$  is 0. Note that since  $\tau_1 = 0$ , we have

$$\sum \tau_i = -(k\omega_1, o^*(v\rho - \rho)) = (v\rho - \rho, -ko_*\omega_1) = (v\rho - \rho, -k\omega_1 + k((\omega_1, \xi)/\xi^2)\xi) = k(v\rho - \rho, \xi)(\omega_1, \xi)/\xi^2.$$

So, we see that the difference of the characters is

$$(v\rho - \rho, c\xi) - (c\xi, \omega_1)k(v\rho - \rho, \xi)(\omega_1, \xi)/\xi^2 = (v\rho - \rho, c\xi)(1 - k(\omega_1, \xi)^2/\xi^2) = z(v).$$

Thus, twisting  $V_{\text{GL}_k}^\tau \otimes V_{\mathbf{L}_{\text{inn}}}^{\nu_v}$  by  $V_{\mathbb{G}_m}^{z(v)}$  we obtain an isomorphism.  $\square$

**Lemma 7.7.** *Let  $\mu = \mu_0 + j\xi + i_*\nu \in \mathbf{B}$ . Then there exists a unique nonincreasing sequence of integers  $\mu_\bullet = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_a \geq \mu_{a+1} = \dots = \mu_k = 0)$  such that*

$$\pi^* V_{\mathbf{L}}^\mu = \varpi^*(V_{\text{GL}_k}^{\mu_\bullet} \otimes V_{\mathbb{G}_m}^{cj\xi^2} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu).$$

*Proof.* Note that  $V_{\mathbf{L}}^\mu = V_{\mathbf{L}}^{\mu_0} \otimes V_{\mathbf{L}}^{j\xi + i_*\nu}$ . Since  $\mu_0 \in \text{Ker } h^* \cap P_{\mathbf{G}}^+$ , we already know from Lemma 7.5 that  $\pi^* V_{\mathbf{L}}^{\mu_0} = \varpi^* V_{\text{GL}_k}^{\mu_\bullet}$  for uniquely determined sequence  $\mu_\bullet = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_a \geq \mu_{a+1} = \dots = \mu_k = 0)$ . So, it remains to express  $\pi^* V_{\mathbf{L}}^{j\xi + i_*\nu}$  as a product of representations of  $\mathbb{G}_m$  and  $\mathbf{L}_{\text{inn}}$ . Since  $i^*(j\xi + i_*\nu) = \nu$ , the  $\mathbf{L}_{\text{inn}}$ -component is  $V_{\mathbf{L}_{\text{inn}}}^\nu$ . On the other hand, the  $\mathbb{G}_m$ -component is  $(c\xi, j\xi + i_*\nu) = cj\xi^2$ .  $\square$

**Proposition 7.8.** *A representation  $\varpi^*(V_{\text{GL}_k}^{\lambda_\bullet} \otimes V_{\mathbb{G}_m}^z \otimes V_{\mathbf{L}_{\text{inn}}}^\nu)$  is isomorphic to a pullback via  $\pi$  of a representation in  $\mathbf{B}$  if and only if  $\nu \in \mathbf{B}^{\text{inn}}$ ,*

$$(40) \quad \sum_{i=1}^a (\lambda_i - \lambda_{i+1})\omega_i \in \mathbf{B}^{\text{out}},$$

and

$$(41) \quad \lambda_{a+1} = \dots = \lambda_k = \frac{cj\xi^2 - z}{k(c\xi, \omega_1)}.$$

*Proof.* Note that by Lemma 7.4 for any  $s \in \mathbb{Z}$  we have

$$\varpi^*(V_{\text{GL}_k}^{(\lambda_1, \dots, \lambda_k)} \otimes V_{\mathbb{G}_m}^z \otimes V_{\mathbf{L}_{\text{inn}}}^\nu) \cong \varpi^*(V_{\text{GL}_k}^{(\lambda_1-s, \dots, \lambda_k-s)} \otimes V_{\mathbb{G}_m}^{z+sk(c\xi, \omega_1)} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu).$$

So, taking  $s = \lambda_k$  and using Lemma 7.7 we deduce  $z + k\lambda_k(c\xi, \omega_1) = cj\xi^2$ . The Proposition follows.  $\square$



**7.2. Proof of the compatibility.** The goal of this section is to prove Theorem 7.1. So we take arbitrary  $\mu = \mu_0 + j\xi + i_*\nu \in \mathbf{B}$ , consider the tensor product  $V_{\mathbf{L}}^\kappa \otimes V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^\mu$  and look at all its irreducible components which are in  $\mathbf{B}$ . For this we pull it back to  $\mathbf{L}_{\text{out}} \times \mathbb{G}_m \times \mathbf{L}_{\text{inn}}$  and use decompositions of Lemmas 7.5, 7.6, and 7.7. We will get

$$\varpi^*(V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}) \bigotimes V_{\mathbb{G}_m}^{z(v)+cj\xi^2} \bigotimes q^*(V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu).$$

By Proposition 7.8 we should take all irreducible components of  $V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu$  which are in  $\mathbf{B}^{\text{inn}}$  and all irreducible components  $V_{\text{GL}_k}^{\lambda\bullet}$  of  $V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}$  which satisfy (40) and (41). Note that (41) shows that  $\lambda_k$  is equal to

$$-\frac{z(v)}{k(c\xi, \omega_1)} = \frac{(\rho - v\rho, \xi)}{(\xi, \omega_1)} \left(1 - k \frac{(\xi, \omega_1)^2}{\xi^2}\right) = \phi(v).$$

So, we conclude that

$$(42) \quad \lambda_{a+1} = \dots = \lambda_k = \phi(v).$$

Let  $S^{\text{inn}} = \mathbf{B}^{\text{inn}}$  and let  $S^{\text{out}}$  be the set of all  $\lambda_\bullet$  satisfying (40) and (42). Then

$$\begin{aligned} \Pi_{\pi^*\mathbf{B}} \varpi^* \left( (V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}) \bigotimes V_{\mathbb{G}_m}^{z(v)+cj\xi^2} \bigotimes (V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu) \right) = \\ = \varpi^* \left( \Pi_{S^{\text{out}}} (V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}) \bigotimes V_{\mathbb{G}_m}^{z(v)+cj\xi^2} \bigotimes \Pi_{S^{\text{inn}}} (V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu) \right). \end{aligned}$$

This means that the pullback via  $\pi$  of the map (38) coincides with the pullback via  $\varpi$  of the tensor product of the maps

$$(43) \quad V_{\text{GL}_k}^{\kappa\bullet+\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet} \rightarrow V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet} \rightarrow V_{\text{GL}_k}^{\kappa\bullet} \otimes \Pi_{S^{\text{out}}} (V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet})$$

and

$$(44) \quad V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu \rightarrow V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu \rightarrow \Pi_{S^{\text{inn}}} (V_{\mathbf{L}_{\text{inn}}}^{\nu_v} \otimes V_{\mathbf{L}_{\text{inn}}}^\nu)$$

(and everything is twisted by  $z(v)+cj\xi^2$ ), and we have to prove that (43) is an  $S^{\text{out}}$ -isomorphism, and (44) is an  $S^{\text{inn}}$ -isomorphism. Note that the second holds automatically, so we can concentrate on the first.

Let  $\tilde{S}^{\text{out}}$  be the set of all  $\lambda_\bullet$  satisfying only (42). We claim that if we replace in (43) the projector  $\Pi_{S^{\text{out}}}$  by  $\Pi_{\tilde{S}^{\text{out}}}$ , then the obtained map

$$(45) \quad V_{\text{GL}_k}^{\kappa\bullet+\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet} \rightarrow V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet} \rightarrow V_{\text{GL}_k}^{\kappa\bullet} \otimes \Pi_{\tilde{S}^{\text{out}}} (V_{\text{GL}_k}^{\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet})$$

will be an  $\tilde{S}^{\text{out}}$ -isomorphism. Indeed, if  $\phi(v)$  is a nonpositive integer then this is Corollary 10.3. If  $\phi(v)$  is not an integer, then  $\tilde{S}^{\text{out}} = \emptyset$ , so any map is an  $\tilde{S}^{\text{out}}$ -isomorphism. Finally, if  $\phi(v)$  is a positive integer then  $v$  is very special, hence we have  $(\mu + v\rho - \rho, \alpha_1 + \dots + \alpha_{k-1}) < \phi(v)$ . This means that  $\mu_1 + \tau_k < \lambda_k$ , so the tensor products  $V_{\text{GL}_k}^{\kappa\bullet+\tau\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}$  and  $V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\mu\bullet}$  contain no terms in  $\tilde{S}^{\text{out}}$ . Thus, after applying  $\Pi_{\tilde{S}^{\text{out}}}$  both the source and the target of the map (43) become zero, hence the map becomes an isomorphism. This finishes the proof that (45) is an  $\tilde{S}^{\text{out}}$ -isomorphism.

It follows that the map (45) is an  $S^{\text{out}}$ -isomorphism. So, to conclude the proof it suffices to check the following property: if  $\lambda_\bullet \in \tilde{S}^{\text{out}}$  and  $\Pi_{S^{\text{out}}} (V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\lambda\bullet}) \neq 0$  then  $\lambda_\bullet \in S^{\text{out}}$ . But all irreducible components of  $V_{\text{GL}_k}^{\kappa\bullet} \otimes V_{\text{GL}_k}^{\lambda\bullet}$  correspond to Young diagrams containing  $(\lambda_1 - \lambda_{a+1}, \lambda_2 - \lambda_{a+1}, \dots, \lambda_a - \lambda_{a+1})$  as a subdiagram, so if one of them is in  $\mathbf{B}^{\text{out}}$  then  $(\lambda_1 - \lambda_{a+1}, \lambda_2 - \lambda_{a+1}, \dots, \lambda_a - \lambda_{a+1})$  is also in  $\mathbf{B}^{\text{out}}$ , since  $\mathbf{B}^{\text{out}}$  is closed with respect to passing to a Young subdiagram.

## 8. EXPLICIT DESCRIPTION OF THE EXCEPTIONAL BLOCKS

In this section we will pass from the abstract description of the block  $B_j$  given in subsections 5.4 and 5.5 to a more explicit description which will be used later to deal with concrete examples. We show in fact that both the inner and the outer parts of the blocks are described by several simple inequalities, numbered by  $\mathbf{W}_{M_j}$ -orbits in the  $\mathbf{W}_{H_j}$ -orbit of the shape  $\delta_{a(j)}$  of the core  $\mathbf{R}_j$ .

Let us fix  $j \in J$ . It will be convenient to write the shape  $\delta = \delta_{a(j)} \in P_{H_j}^+$  of the core  $\mathbf{R}_j = \mathbf{R}_\delta$  in the form

$$(46) \quad \delta = -h_j^* \gamma,$$

where  $\gamma \in P_G$ .

*Remark 8.1.* Since the action of the Weyl group on roots is much better understood than on arbitrary weights (for example, one can use tables of roots), the most convenient choice of  $\gamma$  is the simple root of the vertex of  $D_G$  adjacent to  $D_{H_j}$ . In this case the  $\mathbf{W}_{H_j}$ -orbit of  $\gamma$  is described in Lemma 2.7.

**8.1. The big blocks.** First, we give a description of the block  $B_j$ .

Assume that  $\gamma \in P_G$  and that  $\delta$  is defined by (46) is  $H_j$ -dominant. To unburden the notation we will write  $\mathbf{H}$  for  $H_j$ ,  $h$  for  $h_j$ , and  $\mathbf{M}$  for  $M_j = L \cap H_j$ . Since  $\mathbf{W}_M \subset \mathbf{W}_H$ , the  $\mathbf{W}_H$ -orbit of  $\gamma$  splits into several  $\mathbf{W}_M$ -orbits. We number the orbits by integers  $0, \dots, m$  in such a way that the 0-th orbit is the  $\mathbf{W}_M$ -orbit of  $\gamma$  itself.

In each  $\mathbf{W}_M$ -orbit we have two special elements: the unique  $\mathbf{M}$ -dominant representative  $\gamma_{t+}$  and the unique  $\mathbf{M}$ -antidominant representative  $\gamma_{t-}$  (where  $0 \leq t \leq m$ ). Note that  $\gamma_{0-} = \gamma$ , since we assumed that  $h^* \gamma = -\delta$  is  $\mathbf{H}$ -antidominant. Using these data we can describe the block  $B_j$  more explicitly. We start with the inner part of the block.

**Proposition 8.2.** *We have*

$$(47) \quad B_j^{\text{inn}} = \left\{ \nu \in P_{L^{\text{inn}}}^+ \mid \begin{array}{l} \max\{(i^* \gamma_{t+}, \nu), -(i^* \gamma_{t-}, \nu)\} \leq \frac{1}{2}(h^*(\gamma_{t-} - \gamma), \rho_H) \text{ for all } 0 \leq t \leq m \\ \text{and } j\xi + i_* \nu \in P_L \end{array} \right\}.$$

*Proof.* By definition,  $B_j^{\text{inn}}$  is the set of all  $\nu$  such that  $j\xi + i_* \nu \in P_L$  and  $\rho_H \pm 2w_{L^{\text{inn}}} i_* \nu \in \mathbf{R}_\delta$  for all  $w_{L^{\text{inn}}} \in \mathbf{W}_{L^{\text{inn}}}$ . We start by analyzing the second condition. Substituting the definition 5.1 of the core  $\mathbf{R}_\delta$ , it can be rewritten as

$$B_j^{\text{inn}} = \hat{B}_j^{\text{inn}} = \{\nu \in P_{L^{\text{inn}}}^+ \mid -(w_H h^* \gamma, \rho_H \pm 2w_{L^{\text{inn}}} i_* \nu) \leq -(h^* \gamma, \rho_H)\}.$$

Since  $h^*$  is  $\mathbf{W}_H$ -equivariant, this inequality can be rewritten as

$$\pm(h^* w_H \gamma, 2w_{L^{\text{inn}}} i_* \nu) \leq (h^*(w_H \gamma - \gamma), \rho_H).$$

Note that  $w_H \gamma = w_M \gamma_{t+}$  for appropriate  $w_M \in \mathbf{W}_M$  and  $t \in \{0, 1, \dots, m\}$ . After such a substitution the inequality takes the form

$$\pm(h^* w_M \gamma_{t+}, 2w_{L^{\text{inn}}} i_* \nu) \leq (h^*(w_M \gamma_{t+} - \gamma), \rho_H).$$

Let  $M_{\text{out}} = L_{\text{out}} \cap H$ . Then  $\mathbf{W}_M = \mathbf{W}_{M_{\text{out}}} \times \mathbf{W}_{L^{\text{inn}}}$ . In particular, we can write  $w_M = w_{M_{\text{out}}} w_{L^{\text{inn}}}$  with  $w_{M_{\text{out}}} \in \mathbf{W}_{M_{\text{out}}}$ ,  $w_{L^{\text{inn}}} \in \mathbf{W}_{L^{\text{inn}}}$ . Moreover,  $i_* \nu$  is fixed by  $\mathbf{W}_{M_{\text{out}}}$ , hence the LHS is equal to

$$\pm 2(h^* \gamma_{t+}, w_M^{-1} w_{L^{\text{inn}}} i_* \nu) = \pm 2(h^* \gamma_{t+}, w_{M_{\text{out}}}^{-1} w_{L^{\text{inn}}}^{-1} w_{L^{\text{inn}}} i_* \nu) = \pm 2(h^* \gamma_{t+}, w'_{L^{\text{inn}}} i_* \nu),$$

where  $w'_{L^{\text{inn}}} = w_{L^{\text{inn}}}^{-1} w_{L^{\text{inn}}}$ . Note that  $w'_{L^{\text{inn}}}$  in the LHS runs through  $\mathbf{W}_{L^{\text{inn}}}$  independently of  $w_M$  in the RHS running through  $\mathbf{W}_M$ . Hence, the inequality for all  $w'_{L^{\text{inn}}} \in \mathbf{W}_{L^{\text{inn}}}$ ,  $w_M \in \mathbf{W}_M$  is equivalent to

$$\max_{w'_{L^{\text{inn}}} \in \mathbf{W}_{L^{\text{inn}}}} \{\pm 2(h^* \gamma_{t+}, w'_{L^{\text{inn}}} i_* \nu)\} \leq \min_{w_M \in \mathbf{W}_M} \{(h^*(w_M \gamma_{t+} - \gamma), \rho_H)\}.$$

The expression under the maximum can be rewritten as  $\pm 2((w'_{\mathbf{L}_{\text{inn}}})^{-1}i^*\gamma_{t+}, \nu)$ . Since both  $\nu$  and  $i^*\gamma_{t+}$  are  $\mathbf{L}_{\text{inn}}$ -dominant the expression with “+” sign is maximized on  $w'_{\mathbf{L}_{\text{inn}}} = 1$ , and the expression with “−” sign is maximized on  $(w'_{\mathbf{L}_{\text{inn}}})^{-1}i^*\gamma_{t+} = i^*\gamma_{t-}$ . Thus, the LHS is

$$\max\{2(i^*\gamma_{t+}, \nu), -2(i^*\gamma_{t-}, \nu)\}.$$

Similarly, since  $\rho_{\mathbf{H}}$  is  $\mathbf{M}$ -dominant, the expression in the RHS is minimized on  $w_{\mathbf{M}}\gamma_{t+} = \gamma_{t-}$ . The claim follows.  $\square$

Now let us rewrite more explicitly the definition of the outer part of the block  $\mathbf{B}_j^{\text{out}}$ . Denote by  $\hat{\gamma}_t$  the  $\mathbf{L}_{\text{out}}$ -dominant representative in the  $\mathbf{W}_{\mathbf{L}_{\text{out}}}$ -orbit of  $h_*h^*\gamma_{t+}$ . Also, set

$$(48) \quad \begin{aligned} d_j^{t,+} &:= \max\{(i^*\gamma_{t+}, \nu) \mid \nu \in \mathbf{B}_j^{\text{inn}}\}, \\ d_j^{t,-} &:= -\min\{(i^*\gamma_{t-}, \nu) \mid \nu \in \mathbf{B}_j^{\text{inn}}\}. \end{aligned}$$

**Proposition 8.3.** *We have*

$$(49) \quad \hat{\mathbf{B}}_j^{\text{out}} = \{\lambda \in \text{Ker } h^* \cap P_{\mathbf{G}}^+ \mid (\lambda, \hat{\gamma}_t) + d_j^{t,+} + d_j^{t,-} \leq (\rho_{\mathbf{H}}, \gamma_{t-} - \gamma) \text{ for all } 0 \leq t \leq m\}.$$

*Proof.* Take  $\lambda \in \text{Ker } h^* \cap P_{\mathbf{G}}^+$ . By definition  $\lambda \in \hat{\mathbf{B}}_j^{\text{out}}$  if and only if  $h^*(\rho - w_{\mathbf{L}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu' \in \mathbf{R}_{\delta}$ . By definition of  $\mathbf{R}_{\delta}$  this is equivalent to

$$(h^*(\rho - w_{\mathbf{L}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu', -v_{\mathbf{H}}h^*\gamma) \leq (\rho_{\mathbf{H}}, -h^*\gamma)$$

for all  $\nu, \nu' \in \mathbf{B}_j^{\text{inn}}$ ,  $w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}$ ,  $w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}}$ , and  $v_{\mathbf{H}} \in \mathbf{W}_{\mathbf{H}}$ . Note that  $\mathbf{W}_{\mathbf{L}} = \mathbf{W}_{\mathbf{L}_{\text{out}}} \times \mathbf{W}_{\mathbf{L}_{\text{inn}}}$  and that  $\lambda$  is  $\mathbf{W}_{\mathbf{L}_{\text{inn}}}$ -invariant. So we can rewrite the above condition as

$$(h^*(\rho - w_{\mathbf{L}_{\text{out}}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu', -v_{\mathbf{H}}h^*\gamma) \leq (\rho_{\mathbf{H}}, -h^*\gamma)$$

Since  $h^*$  is  $\mathbf{W}_{\mathbf{H}}$ -equivariant we have  $v_{\mathbf{H}}h^*\gamma = h^*(v_{\mathbf{H}}\gamma)$ . Further, each weight  $v_{\mathbf{H}}\gamma$  can be written as  $v_{\mathbf{M}}\gamma_{t+}$  for some  $0 \leq t \leq m$  and  $v_{\mathbf{M}} \in \mathbf{W}_{\mathbf{M}}$ . This allows to rewrite the condition as

$$(h^*(\rho - w_{\mathbf{L}_{\text{out}}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu', -v_{\mathbf{M}}h^*\gamma_{t+}) \leq (\rho_{\mathbf{H}}, -h^*\gamma)$$

for all  $\nu, \nu' \in \mathbf{B}_j^{\text{inn}}$ ,  $w_{\mathbf{L}_{\text{out}}} \in \mathbf{W}_{\mathbf{L}_{\text{out}}}$ ,  $w_{\mathbf{L}_{\text{inn}}}, w'_{\mathbf{L}_{\text{inn}}} \in \mathbf{W}_{\mathbf{L}_{\text{inn}}}$ ,  $v_{\mathbf{M}} \in \mathbf{W}_{\mathbf{M}}$ , and all  $t$ ,  $0 \leq t \leq m$ . Now recall that  $h^*\rho = \rho_{\mathbf{H}}$  and move it from the LHS to the RHS:

$$(h^*(-w_{\mathbf{L}_{\text{out}}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu', -v_{\mathbf{M}}h^*\gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Now, writing  $v_{\mathbf{M}} = v_{\mathbf{M}_{\text{out}}}v_{\mathbf{L}_{\text{inn}}}$  in the LHS, taking into account that  $h^*$  is  $\mathbf{W}_{\mathbf{M}}$ -equivariant, and substituting  $v_{\mathbf{M}_{\text{out}}}^{-1}w_{\mathbf{L}_{\text{out}}}$  with  $w_{\mathbf{L}_{\text{out}}}$ ,  $v_{\mathbf{L}_{\text{inn}}}^{-1}w_{\mathbf{L}_{\text{inn}}}$  with  $w_{\mathbf{L}_{\text{inn}}}$ , and  $v_{\mathbf{L}_{\text{inn}}}^{-1}w'_{\mathbf{L}_{\text{inn}}}$  with  $w'_{\mathbf{L}_{\text{inn}}}$  we rewrite the condition as

$$(h^*(-w_{\mathbf{L}_{\text{out}}}\lambda) - w_{\mathbf{L}_{\text{inn}}}i_*\nu + w'_{\mathbf{L}_{\text{inn}}}i_*\nu', -h^*\gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Finally, using the adjunction of  $h^*$  and  $h_*$  and of  $i^*$  and  $i_*$  we rewrite this as

$$(w_{\mathbf{L}_{\text{out}}}\lambda, h_*h^*\gamma_{t+}) + (w_{\mathbf{L}_{\text{inn}}}\nu, i^*\gamma_{t+}) + (-w'_{\mathbf{L}_{\text{inn}}}\nu', i^*\gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Note that each term on both sides contains an action of a Weyl group element, and these elements run through the corresponding Weyl groups independently. Therefore, one can replace each summand by its maximum (in the LHS) or minimum (in the RHS) to obtain an equivalent inequality.

The maximums of the second and the third summands in the LHS are given by  $d_j^{t,\pm}$  by definition. The first summand can be rewritten as  $(\lambda, w_{\mathbf{L}_{\text{out}}}^{-1}h_*h^*\gamma_{t+})$  and since  $\lambda$  is  $\mathbf{L}_{\text{out}}$ -dominant to achieve the maximum one should choose  $w_{\mathbf{L}_{\text{out}}}^{-1}$  in such a way that the corresponding weight is also  $\mathbf{L}_{\text{out}}$ -dominant. By definition, it is  $\hat{\gamma}_t$ , hence the maximum of the first summand is  $(\lambda, \hat{\gamma}_t)$ . Finally, as in Proposition 8.2, we obtain that the minimum in the RHS is equal to  $(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$ . Combining all of this together we obtain the result.  $\square$

**8.2. The small blocks.** Now we will give a description of the blocks  $\bar{B}_j$ .

Take  $j, j' \in J$  and assume that  $j' < j$ . As before we write  $\mathbf{H}$  for  $\mathbf{H}_j$ ,  $h$  for  $h_j$ , and  $\mathbf{M}$  for  $\mathbf{M}_j = \mathbf{L} \cap \mathbf{H}_j$ . In addition, we will write  $\mathbf{H}'$  for  $\mathbf{H}_{j'}$ ,  $h'$  for  $h_{j'}$ , and  $\mathbf{M}'$  for  $\mathbf{M}_{j'} = \mathbf{L} \cap \mathbf{H}_{j'}$ . Similarly we denote by  $\gamma$  and  $\gamma'$  the weights such that  $\delta = -h^*\gamma$  and  $\delta' = -(h')^*\gamma'$  are the shapes of the corresponding cores. We number the orbits of  $\mathbf{W}_{\mathbf{M}'}$  on  $\mathbf{W}_{\mathbf{H}'}\gamma'$  from 0 to  $m'$ , and we denote by  $\gamma'_{t\pm}$  the  $\mathbf{M}'$ -dominant and antidominant representatives of these orbits.

The proof of the next two results is analogous to that of Propositions 8.2 and 8.3.

**Proposition 8.4.** *The inner part of the block  $\bar{B}_j$  can be described by the following system of inequalities*

$$(50) \quad \bar{B}_j^{\text{inn}} = \left\{ \nu \in B_j^{\text{inn}} \mid \begin{array}{l} (j-j')((h')^*\xi, (h')^*\gamma'_{t+}) + (i^*\gamma'_{t+}, \nu) + \bar{d}_{j'}^{t,-} < (\rho_{\mathbf{H}'}, (h')^*(\gamma'_{t-} - \gamma')) \\ \text{for all } j' < j \text{ and for all } 0 \leq t \leq m' \end{array} \right\},$$

where for  $j' < j$

$$(51) \quad \bar{d}_{j'}^{t,-} := -\min\{(i^*\gamma'_{t-}, \nu') \mid \nu' \in \bar{B}_{j'}^{\text{inn}}\}.$$

**Proposition 8.5.** *The outer part of the block  $\bar{B}_j$  can be described by the following system of inequalities*

$$(52) \quad \bar{B}_j^{\text{out}} = \left\{ \lambda \in B_j^{\text{out}} \mid \begin{array}{l} (\lambda, \hat{\gamma}'_t) + (j-j')((h')^*\xi, (h')^*\gamma'_{t+}) + \bar{d}_{j',j}^{t,+} + \bar{d}_{j'}^{t,-} < (\rho_{\mathbf{H}'}, (h')^*(\gamma'_{t-} - \gamma')) \\ \text{for all } j' < j \text{ and for all } 0 \leq t \leq m' \end{array} \right\},$$

where

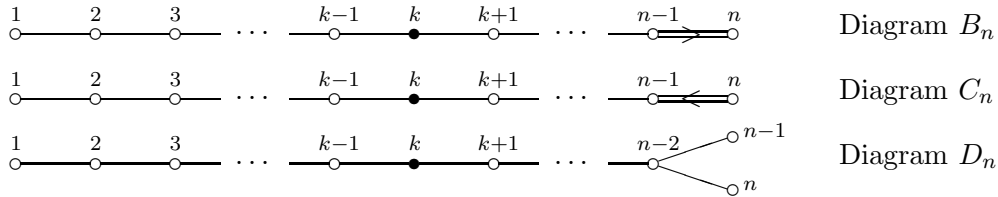
$$(53) \quad \bar{d}_{j',j}^{t,+} := \max\{(i^*\gamma'_{t+}, \nu) \mid \nu \in \bar{B}_j^{\text{inn}}\}$$

and  $\hat{\gamma}'_t$  is the  $\mathbf{L}_{\text{out}}$ -dominant representative in  $\mathbf{W}_{\mathbf{L}_{\text{out}}}h'_*(h')^*\gamma'_{t+}$ .

## 9. EXPLICIT COLLECTIONS FOR CLASSICAL GROUPS

The construction of the previous section allows to construct a (conjecturally full) exceptional collection for isotropic Grassmannians (in types B, C and D), and many interesting collections in type A.

So, assume that  $\mathbf{G}$  is of type B, C or D and consider the standard numbering of the vertices.



To treat these cases simultaneously it is convenient to denote

$$(54) \quad e = \begin{cases} 1/2, & \text{if } \mathbf{G} \text{ is of type } B, \\ 1, & \text{if } \mathbf{G} \text{ is of type } C, \\ 0, & \text{if } \mathbf{G} \text{ is of type } D. \end{cases}$$

Then the weight lattice  $P_{\mathbf{G}}$  can be identified with the sublattice of  $\mathbb{Q}^n$  spanned by  $\omega_i$  ( $1 \leq i \leq n$ ) with

$$\begin{aligned} \omega_i^{B,C,D} &= (\underbrace{1, 1, \dots, 1}_i, \underbrace{0, 0, \dots, 0}_{n-i}), \quad 1 \leq i \leq n-2+2e, \\ \omega_n^{B,D} &= (1/2, 1/2, \dots, 1/2, 1/2), \\ \omega_{n-1}^D &= (1/2, 1/2, \dots, 1/2, -1/2). \end{aligned}$$

Let  $k$  be the number of the vertex of the Dynkin diagram of  $\mathbf{G}$  corresponding to the maximal parabolic subgroup  $\mathbf{P}$ , so that  $\xi = \omega_k$ .

9.1. **Isotropic Grassmannians.** First, we assume that

$$k \leq n + 2e - 2.$$

In other words,  $k \leq n - 1$  for type  $B$ ,  $k \leq n$  for type  $C$  and  $k \leq n - 2$  for type  $D$ . Then

$$X := \mathbf{G}/\mathbf{P} = \begin{cases} \text{OGr}(k, 2n+1), & k \leq n-1 \text{ (}\mathbf{G} \text{ is of type } B_n\text{)} \\ \text{SGr}(k, 2n), & k \leq n \text{ (}\mathbf{G} \text{ is of type } C_n\text{)} \\ \text{OGr}(k, 2n), & k \leq n-2 \text{ (}\mathbf{G} \text{ is of type } D_n\text{)} \end{cases}$$

Let  $D_{\text{out}}$  be the component containing vertices from 1 to  $k-1$ . Then  $b = k-1$  and  $D_{\text{inn}}$  is the component containing vertices from  $k+1$  to  $n$ . Note that  $i^*$  is the projection onto the last  $n-k$  coordinates while  $o^*$  is the projection onto the first  $k-1$  coordinates. Let  $\varepsilon_1, \dots, \varepsilon_n$  be the standard basis in  $P_{\mathbf{G}} = \mathbb{Q}^n$ . Then the simple roots are

$$\begin{aligned} \alpha_i^{B,C,D} &= \varepsilon_i - \varepsilon_{i+1}, & 1 \leq i \leq n-1, \\ \alpha_n^B &= \varepsilon_n, & \alpha_n^C = 2\varepsilon_n, & \alpha_n^D = \varepsilon_{n-1} + \varepsilon_n. \end{aligned}$$

Note also that

$$\rho = (n+e-1, n+e-2, \dots, e),$$

thus  $(\rho, \varepsilon_i) = n+e-i$ .

Now take any  $a \leq k-1$ . Then the projection  $h_a^*$  is the projection onto the last  $n-a$  coordinates (it kills all  $\varepsilon_i$  with  $i \leq a$ ). The simple root corresponding to  $\mathbf{P}$  is  $\beta = \varepsilon_k - \varepsilon_{k+1}$ , so the maximal root of  $\mathbf{H}_a$  with the coefficient of  $\beta$  equal to 1 is  $\bar{\beta}_a = \varepsilon_{a+1} + \varepsilon_{k+1}$ , so by Lemma 2.18 the index of the Grassmannian  $\mathbf{H}_a/(\mathbf{H}_a \cap \mathbf{P})$  is

$$r_a = (\rho, \beta + \bar{\beta}_a)/(\xi, \beta) = 2n + 2e - a - k - 1.$$

In particular, we see that when  $a$  decreases by 1,  $r_a$  increases by 1. Also, note that  $r_{k-1} = 2n + 2e - 2k$ , while

$$r = r_0 = 2n + 2e - k - 1.$$

Further, the weight  $\theta$  defined by (25) in this case is

$$\theta = (\underbrace{0, 0, \dots, 0}_{k-1}, \underbrace{1, 0, 0, \dots, 0}_{n-k}).$$

It follows that  $(\theta, P_{\mathbf{L}}) = \frac{1}{2}\mathbb{Z}$  if  $\mathbf{G}$  is of type  $B$  or  $D$  and  $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$  if  $\mathbf{G}$  is of type  $C$  and

$$J = \begin{cases} \frac{1}{2}\mathbb{Z} \cap [0, 2n - k - 1/2], & \text{if } \mathbf{G} \text{ is of type } B \\ \mathbb{Z} \cap [0, 2n - k], & \text{if } \mathbf{G} \text{ is of type } C \\ \frac{1}{2}\mathbb{Z} \cap [0, 2n - k - 3/2], & \text{if } \mathbf{G} \text{ is of type } D \end{cases}$$

Applying (26) we conclude that

$$a(j) = \begin{cases} \lfloor j \rfloor, & \text{if } j < k \\ k-1, & \text{if } j \geq k \end{cases}$$

Now we are going to apply Propositions 8.2, 8.3, 8.4 and 8.5. We take

$$\gamma_a = \alpha_a = \varepsilon_a - \varepsilon_{a+1}.$$

Note that  $\mathbf{W}_{\mathbf{H}_a}$  acts by permutations of the last  $n-a$  coordinates and changes of signs of the coordinates (in case of type  $D$  by pairwise changes of signs), while  $\mathbf{W}_{\mathbf{M}_a}$  acts by permuting coordinates from  $a+1$  to  $k$  and from  $k+1$  to  $n$  separately and (pairwise) changes of signs only of last  $n-k$  coordinates. Thus, the  $\mathbf{W}_{\mathbf{H}_a}$ -orbit of  $\gamma_a$  consists of all vectors  $\varepsilon_a \pm \varepsilon_i$ ,  $a+1 \leq i \leq n$ , and it splits into three  $\mathbf{W}_{\mathbf{M}_a}$ -orbits:

$$\{\varepsilon_a - \varepsilon_i\}_{a+1 \leq i \leq k}, \quad \{\varepsilon_a \pm \varepsilon_i\}_{k+1 \leq i \leq n}, \quad \text{and} \quad \{\varepsilon_a + \varepsilon_i\}_{a+1 \leq i \leq k}.$$

Thus, using the notation of section 8 we have  $m = 2$  (unless  $\mathbf{G}$  has type  $C$  and  $k = n$  in which case the second orbit is empty and so  $m = 1$ ), and the characteristic weights and quantities from section 8 are given by the following table:

$t$	$\gamma_{t-}$	$(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$	$\gamma_{t+}$	$h_a^* \gamma_{t+}$	$\hat{\gamma}_t$	$(h_a^* \xi, h_a^* \gamma_{t+})$	$i^* \gamma_{t+}$	$i^* \gamma_{t-}$
0	$\varepsilon_a - \varepsilon_{a+1}$	0	$\varepsilon_a - \varepsilon_k$	$-\varepsilon_k$	$-\varepsilon_k$	-1	0	0
1	$\varepsilon_a - \varepsilon_{k+1}$	$k - a$	$\varepsilon_a + \varepsilon_{k+1}$	$\varepsilon_{k+1}$	$\varepsilon_{k+1}$	0	$\varepsilon_{k+1}$	$-\varepsilon_{k+1}$
2	$\varepsilon_a + \varepsilon_k$	$2n + 2e - a - k - 1$	$\varepsilon_a + \varepsilon_{a+1}$	$\varepsilon_{a+1}$	$\varepsilon_1$	1	0	0

(if  $\mathbf{G}$  has type  $C$  and  $k = n$  then the line  $t = 1$  should be omitted).

Applying Proposition 8.2 we obtain the following description of  $\mathbf{B}_j^{\text{inn}}$ :

$$\mathbf{B}_j^{\text{inn}} = \{(\nu_{k+1}, \dots, \nu_n) \in P_{\mathbf{L}^{\text{inn}}}^+ \mid 2\nu_{k+1} \leq k - a(j) \text{ and } \nu_i \equiv j \pmod{\mathbb{Z}}\}.$$

Further, we apply (48) and compute

$$d_j^{1,\pm} = \{j\} + \lfloor (k - a(j))/2 - \{j\} \rfloor,$$

(where  $\{-\}$  stands for the fractional part) and for other values of  $t$  we have  $d_j^{t,\pm} = 0$ .

Now we can describe  $\mathbf{B}_j^{\text{out}}$ . Note that  $(\hat{\gamma}_t, \text{Ker } h_a^*) = 0$  unless  $t = 2$ . So, for  $t = 0$  Proposition 8.3 gives an empty condition and for  $t = 1$  we obtain the condition that  $d_j^{1,+} + d_j^{1,-} \leq k - a$  which holds by the definition of  $d_j^{1,\pm}$ . Finally, the condition for  $t = 2$  gives

$$\mathbf{B}_j^{\text{out}} = \{(\lambda_1, \dots, \lambda_{a(j)}, 0, \dots, 0) \mid 2n + 2e - a(j) - k - 1 \geq \lambda_1 \geq \dots \geq \lambda_{a(j)} \geq 0\}.$$

Note that the set  $\mathbf{B}_j^{\text{out}}$  is the set of Young diagrams inscribed into the rectangle  $a(j) \times (2n + 2e - a(j) - k - 1)$ , hence it is closed under taking subdiagrams. Thus, the second condition of Theorem 7.1 is satisfied. Since there are no very special elements by Lemma 5.8 the first condition is satisfied as well, so Theorem applies and we conclude that the block

$$\mathbf{B}_j = \left\{ (\lambda_1, \dots, \lambda_n) \in P_{\mathbf{L}}^+ \mid \begin{array}{l} 2n + 2e + j - a(j) - k - 1 \geq \lambda_1 \geq \dots \geq \lambda_{a(j)} \geq j = \lambda_{a(j)+1} = \dots = \lambda_k, \\ (k - a(j))/2 \geq \lambda_{k+1} \geq \dots \geq \lambda_n, \\ \lambda_1, \dots, \lambda_n \equiv j \pmod{\mathbb{Z}} \end{array} \right\}$$

is exceptional.

Now we are going to apply Proposition 8.4. First, let us show that

$$(55) \quad \bar{\mathbf{B}}_j^{\text{inn}} = \mathbf{B}_j^{\text{inn}} \quad \text{for } j < k$$

and  $\bar{d}_j^{1,-} = d_j^{1,-} = \{j\} + \lfloor (k - a(j))/2 - \{j\} \rfloor$ . For this we can use induction in  $j$ . The base of induction,  $j = 0$  is clear. Assume that for all  $j' < j$  the statement is proved. Then by Proposition 8.4, the additional condition defining  $\bar{\mathbf{B}}_j^{\text{inn}}$  is

$$\nu_{k+1} + \{j'\} + \lfloor (k - a(j'))/2 - \{j'\} \rfloor < k - a(j').$$

We claim that this condition is always satisfied for  $\nu \in \mathbf{B}_j^{\text{inn}}$ . Indeed, we have

$$(56) \quad \begin{aligned} \{j\} + \lfloor (k - a(j))/2 - \{j\} \rfloor + \{j'\} + \lfloor (k - a(j'))/2 - \{j'\} \rfloor &\leq \\ (k - a(j))/2 + (k - a(j'))/2 &= k - (a(j) + a(j'))/2 \leq k - a(j'), \end{aligned}$$

and the equality is possible only for if  $a(j) = a(j')$  and both  $(k - a(j))/2 - \{j\}$  and  $(k - a(j))/2 - \{j'\}$  are integers. But for  $j', j < k$  one has  $a(j) = \lfloor j \rfloor$ , so the first shows that the integer parts of  $j$  and  $j'$  are equal, while the second shows that the difference  $j - j'$  is integer. This is possible only if  $j = j'$ , which is a contradiction. Hence, one of the inequalities in (56) is strict as we claimed. This finishes the proof of (55).

Now let us check that

$$\begin{aligned}\bar{\mathbf{B}}_j^{\text{inn}} &= 0, \quad \text{for integer } j \geq k \text{ and} \\ \bar{\mathbf{B}}_j^{\text{inn}} &= \emptyset, \quad \text{for half-integer } j \geq k.\end{aligned}$$

Indeed, if  $j$  is half-integer take  $j' = k - 1/2$ . Then  $\bar{d}_{j'}^{1,-} = \{j'\} + \lfloor (k - a(j'))/2 - \{j'\} \rfloor = 1/2 + \lfloor 1/2 - 1/2 \rfloor = 1/2$ , so the inequality defining  $\bar{\mathbf{B}}_j^{\text{inn}} \subset \mathbf{B}_j^{\text{inn}}$  is

$$\nu_{k+1} + 1/2 < 1.$$

On the other hand,  $\nu_{k+1}$  should be a nonnegative half-integer, so we conclude that  $\bar{\mathbf{B}}_j^{\text{inn}} = \emptyset$ . For an integer  $j \geq k$  we note that already  $\mathbf{B}_j^{\text{inn}} = 0$ , so we only have to check that the inequality (50) is satisfied for  $\nu = 0$ . Indeed, if  $j' < k$  then  $a(j') \leq k - 1$ , hence

$$0 + \bar{d}_{j'}^{1,-} = \{j'\} + \lfloor (k - a(j'))/2 - \{j'\} \rfloor \leq \{j'\} + (k - a(j'))/2 - \{j'\} = (k - a(j'))/2 < k - a(j').$$

Further, if  $j' \geq k$  is a half-integer then as we already know  $\bar{\mathbf{B}}_{j'}^{\text{inn}}$  is empty, so  $\bar{d}_{j'}^{1,-} = -\infty$  and so we do not have a restriction on  $\nu$ . Finally, if  $j' \geq k$  is integer then by induction hypothesis we have  $\bar{d}_{j'}^{1,-} = 0$  while  $a(j') = k - 1$ , so the inequality  $\nu_{k+1} + \bar{d}_{j'}^{1,-} < k - a(j')$  holds in this case.

Now let us describe the outer parts of the blocks,  $\bar{\mathbf{B}}_j^{\text{out}}$ . The inequality (52) gives

$$\lambda_1 + j - j' < 2n + 2e - a(j') - k - 1.$$

It can be rewritten as

$$\lambda_1 < 2n + 2e - j - k - 1 + (j' - a(j')).$$

Since this inequality should hold for all  $j' < j$  we can replace the last summand by its minimum, which is equal to 0. So, we conclude that the defining inequality of  $\bar{\mathbf{B}}_j^{\text{out}}$  is  $\lambda_1 < 2n + 2e - j - k - 1$ . Since  $\lambda_1$  should be integer this is equivalent to  $\lambda_1 \leq 2n + 2e - \lfloor j \rfloor - k - 2$ .

Now we can write down the obtained answer. We denote by  $\mathcal{A}_j$  the subcategory in  $\mathcal{D}(X)$  corresponding to the block  $\bar{\mathbf{B}}_j$ .

**Theorem 9.1.** *Let  $\mathbf{G}$  be of type  $B$  or  $D$ . Assume that  $k \leq n - 1$  for type  $B$  and  $k \leq n - 2$  for type  $D$ . For each integer  $t$ ,  $0 \leq t \leq k - 1$ , consider the subcategories  $\mathcal{A}_t$  and  $\mathcal{A}_{t+1/2}$  in  $\mathcal{D}(X)$  defined by*

$$\begin{aligned}\mathcal{A}_t &= \left\langle \mathcal{E}^\lambda \left| \begin{array}{l} 2n + 2e - k - 2 \geq \lambda_1 \geq \dots \geq \lambda_t \geq t = \lambda_{t+1} = \dots = \lambda_k, \\ (k - t)/2 \geq \lambda_{k+1} \geq \dots \geq \lambda_n \geq (2e - 1)\lambda_{n-1}, \end{array} \right. \lambda_i \in \mathbb{Z} \right\rangle, \\ \mathcal{A}_{t+1/2} &= \left\langle \mathcal{E}^\lambda \left| \begin{array}{l} 2n + 2e - k - 3/2 \geq \lambda_1 \geq \dots \geq \lambda_t \geq t + 1/2 = \lambda_{t+1} = \dots = \lambda_k, \\ (k - t)/2 \geq \lambda_{k+1} \geq \dots \geq \lambda_n \geq (2e - 1)\lambda_{n-1}, \end{array} \right. \lambda_i \in 1/2 + \mathbb{Z} \right\rangle,\end{aligned}$$

where  $e$  is defined by (54). Also, for each integer  $t$ ,  $k \leq t \leq 2n + 2e - k - 2$ , consider the subcategory

$$\mathcal{A}_t = \left\langle \mathcal{E}^\lambda \left| \begin{array}{l} 2n + 2e - k - 2 \geq \lambda_1 \geq \dots \geq \lambda_{k-1} \geq \lambda_k = t, \\ \lambda_{k+1} = \dots = \lambda_n = 0, \end{array} \right. \lambda_i \in \mathbb{Z} \right\rangle.$$

Then the collection of subcategories

$$\mathcal{A}_0, \mathcal{A}_{1/2}, \mathcal{A}_1, \mathcal{A}_{3/2}, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k-1/2}, \mathcal{A}_k, \mathcal{A}_{k+1}, \dots, \mathcal{A}_{2n+2e-k-2}$$

is semiorthogonal, and each subcategory is generated by an exceptional collection.

**Theorem 9.2.** Assume  $\mathbf{G}$  is of type  $C$  and  $k \leq n$ . Consider the following subcategories in  $\mathcal{D}(X)$  indexed by integers  $t = 0, \dots, 2n - k$ :

$$\begin{aligned} \mathcal{A}_t &= \left\langle \mathcal{E}^\lambda \left| \begin{array}{l} 2n - k \geq \lambda_1 \geq \dots \geq \lambda_t \geq t = \lambda_{t+1} = \dots = \lambda_k, \\ \lfloor (k - t)/2 \rfloor \geq \lambda_{k+1} \geq \dots \geq \lambda_n \geq 0, \end{array} \right. \right\rangle, \quad \text{for } t \leq k - 1 \\ \mathcal{A}_t &= \left\langle \mathcal{E}^\lambda \left| \begin{array}{l} 2n - k \geq \lambda_1 \geq \dots \geq \lambda_{k-1} \geq \lambda_k = t, \\ \lambda_{k+1} = \dots = \lambda_n = 0, \end{array} \right. \right\rangle, \quad \text{for } t \geq k. \end{aligned}$$

Then the collection of subcategories

$$\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{2n-k},$$

is semiorthogonal, and each subcategory is generated by an exceptional collection.

**9.2. Orthogonal maximal isotropic Grassmannians.** Note that if  $\mathbf{G}$  is of type  $D$  and  $k = n - 1$  or  $k = n$  then the Grassmannian  $\mathbf{G}/\mathbf{P}$  is isomorphic to the Grassmannian of type  $B_{n-1}$  with  $k = n - 1$ . Thus, the only remaining case with  $\mathbf{G}$  classical is when  $\mathbf{G}$  is of type  $B_n$  and  $k = n$ , which will be considered presently. Note that in this case

$$X = \mathbf{G}/\mathbf{P} = \text{OGr}(n, 2n + 1).$$

As before we take  $D_{\text{out}}$  to be the component containing vertices from 1 to  $n - 1$ , and thus  $D_{\text{inn}} = \emptyset$ . Further,  $\beta = \varepsilon_n$ , so  $\bar{\beta}_a = \varepsilon_{a+1}$  and

$$r_a = (\rho, \beta + \bar{\beta}_a) / (\xi, \beta) = 2n - 2a.$$

Hence, when  $a$  increases by 1, the index decreases by 2. In particular,

$$r = r_0 = 2n.$$

The weight  $\theta$  defined by (25) is

$$\theta = (0, 0, \dots, 0, 2),$$

hence  $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$  and

$$J = \mathbb{Z} \cap [0, 2n - 1].$$

Applying (26) we deduce that

$$a(j) = \lfloor j/2 \rfloor.$$

As before we take  $\gamma_a = \alpha_a = \varepsilon_a - \varepsilon_{a+1}$ . Note that  $\mathbf{W}_{\mathbf{H}_a}$  acts by permutations of the last  $n - a$  coordinates and by changes of signs of the coordinates, while  $\mathbf{W}_{\mathbf{M}_a}$  acts just by permutations. Thus, the  $\mathbf{W}_{\mathbf{H}_a}$  orbit of  $\gamma_a$  consists of all vectors  $\varepsilon_a \pm \varepsilon_i$ ,  $a + 1 \leq i \leq n$ , and it splits into two  $\mathbf{W}_{\mathbf{M}_a}$ -orbits:

$$\{\varepsilon_a - \varepsilon_i\}_{a+1 \leq i \leq n} \quad \text{and} \quad \{\varepsilon_a + \varepsilon_i\}_{a+1 \leq i \leq n}.$$

Thus, using the notation of section 8 we have  $m = 1$  and

$t$	$\gamma_{t-}$	$(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$	$\gamma_{t+}$	$h_a^* \gamma_{t+}$	$\hat{\gamma}_t$	$(h_a^* \xi, h_a^* \gamma_{t+})$
0	$\varepsilon_a - \varepsilon_{a+1}$	0	$\varepsilon_a - \varepsilon_n$	$-\varepsilon_n$	$-\varepsilon_n$	$-1/2$
1	$\varepsilon_a + \varepsilon_n$	$n - a$	$\varepsilon_a + \varepsilon_{a+1}$	$\varepsilon_{a+1}$	$\varepsilon_1$	$1/2$

Since  $P_{\mathbf{L}_{\text{inn}}} = 0$  and  $a(j) < k = n$  for all  $j \in J$ , we have

$$B_j^{\text{inn}} = 0 \quad \text{for all } j \in J.$$

In particular,  $d_j^{t,\pm} = 0$  and thus

$$B_j^{\text{out}} = \{n - a(j) \geq \lambda_1 \geq \dots \geq \lambda_{a(j)} \geq 0\}.$$



Note that this is the set of Young diagrams inscribed into the rectangle  $a(j) \times (n - a(j))$ , hence it is closed under taking subdiagrams. Thus, the second condition of Theorem 7.1 is satisfied. Since there are no very special elements by Lemma 5.8, the first condition is satisfied as well, so Theorem applies and we conclude that the block

$$B_j = B_j^{\text{out}} + j\xi = \left\{ (\lambda_1, \dots, \lambda_n) \mid \begin{array}{l} n + j/2 - a(j) \geq \lambda_1 \geq \dots \geq \lambda_{a(j)} \geq j/2 = \lambda_{a(j)+1} = \dots = \lambda_n, \\ \lambda_i \equiv j/2 \pmod{\mathbb{Z}} \end{array} \right\}$$

is exceptional.

On the other hand, the condition (52) gives  $\lambda_1 + (j - j')/2 < n - a(j') = n - \lfloor j'/2 \rfloor$ . It can be rewritten as

$$\lambda_1 < n - j/2 + \{j'/2\}.$$

Since this should be satisfied for all  $j' < j$ , we conclude that  $\lambda_1 < n - j/2$ . On the other hand,  $\lambda_1$  should be an integer, so we obtain  $\lambda_1 \leq n - 1 - \lfloor j/2 \rfloor$ .

Now we can write down the obtained answer.

**Theorem 9.3.** *Assume  $\mathbf{G}$  is of type  $B_n$  and  $k = n$ . Consider the following subcategories in  $\mathcal{D}(X)$  (where  $t$  is a nonnegative integer):*

$$\begin{aligned} \mathcal{A}_{2t} &= \langle \mathcal{E}^\lambda \mid n - 1 \geq \lambda_1 \geq \dots \geq \lambda_t \geq t = \lambda_{t+1} = \dots = \lambda_n, \quad \lambda_i \in \mathbb{Z} \rangle, \\ \mathcal{A}_{2t+1} &= \langle \mathcal{E}^\lambda \mid n - 1/2 \geq \lambda_1 \geq \dots \geq \lambda_t \geq t + 1/2 = \lambda_{t+1} = \dots = \lambda_n, \quad \lambda_i \in 1/2 + \mathbb{Z} \rangle. \end{aligned}$$

Then the following collection of subcategories

$$\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{2n-1},$$

is semiorthogonal, and each subcategory is generated by an exceptional collection.

**9.3. Purity for maximal isotropic Grassmannians.** Recall that for an exceptional block  $B$  the exceptional collection  $(\mathcal{E}^\lambda)_{\lambda \in B}$  is strong if and only if it consists of vector bundles (see Proposition 4.2). Using the explicit form of the blocks we can check that this is true in the case of maximal isotropic Grassmannians (symplectic or orthogonal).

**Theorem 9.4.** *The exceptional collections of Theorem 9.2 for  $k = n$  and of Theorem 9.3 consist of vector bundles.*

*Proof.* By Proposition 4.3, it is enough to check that for each of the blocks  $B$  appearing in the collection the subquiver  $\mathcal{Q}_B \subset \mathcal{Q}$  contains entirely any path that starts and ends in  $\mathcal{Q}_B$ .

First, let us consider the case when  $\mathbf{G}$  is of type  $C_n$  and  $k = n$  (so  $\mathbf{G}/\mathbf{P}$  is the Lagrangian Grassmannian  $\text{SGr}(n, 2n)$ ). In this case  $\mathbf{L} = \text{GL}_n$ , so the quiver  $\mathcal{Q}$  has vertices numbered by dominant weights of  $\text{GL}_n$  and there is an arrow  $\lambda \rightarrow \mu$  if and only if

$$\text{Hom}_{\text{GL}_n}(V^\mu, V^\lambda \otimes (V^{2\omega_1})^\vee) = \text{Hom}_{\text{GL}_n}(V^\mu \otimes V^{2\omega_1}, V^\lambda) \neq 0.$$

Thus, if  $\mu$  corresponds to a Young diagram then so does  $\lambda$  and  $\mu$  is contained in  $\lambda$  as a subdiagram. Since all the blocks consist of Young diagrams and are closed under passing to subdiagrams, this implies that they satisfy our condition on paths.

In the case when  $\mathbf{G}$  is of type  $B_n$  and  $k = n$  the Levi group  $\mathbf{L}$  is a twofold covering of  $\text{GL}_n$ . If  $j$  is integer then all  $\lambda$  and  $\mu$  from this block are restricted from  $\text{GL}_n$  and the arrow  $\lambda \rightarrow \mu$  in  $\mathcal{Q}$  exists if and only if

$$\text{Hom}_{\text{GL}_n}(V^\mu \otimes V^{\omega_1}, V^\lambda) \neq 0,$$

so the above argument shows that the block  $B_j$  satisfies the condition on paths. If  $j$  is half-integer then  $B_j = B_{j-1/2} + \xi$ , and since the twist by  $\xi$  is an autoequivalence, we conclude that the block  $B_j$  satisfies the condition on paths as well.  $\square$

**Example 9.5.** Assume that  $\mathbf{G}$  is of type  $C_4$  and  $k = 3$ , i.e.  $X = \mathbf{G}/\mathbf{P} = \text{SGr}(3, 8)$ , and take the block

$$\mathbf{B}_1 = \{5 \geq \lambda_1 \geq 1 = \lambda_2 = \lambda_3, 1 \geq \lambda_4 \geq 0\}.$$

Note also that  $\mathbf{L} = \text{GL}_3 \times \text{SL}_2$  and  $V_{\mathbf{L}}^{-\beta} = V_{\mathbf{L}}^{0,0,-1;1}$ . In particular, we have a path in the quiver  $\mathcal{Q}$

$$(3, 1, 1; 1) \rightarrow (2, 1, 1; 2) \rightarrow (1, 1, 1; 1)$$

which starts and ends in the block  $\mathbf{B}_1$ , while its second vertex is not in the block. So, the assumption of Proposition 4.3 does not hold. On the other hand, the assumption of Proposition 4.4(i) is not satisfied as well. Indeed, if  $\lambda = (4, 1, 1; 0)$  and  $\mu = (1, 1, 1; 1)$  and  $v = s_3 s_4 \in \text{SR}_{\mathbf{G}}^{\mathbf{L}}$  then  $v\rho - \rho = (0, 0, -3; 1)$  hence  $V_{\mathbf{L}}^{\mu} \subset V_{\mathbf{L}}^{\lambda} \otimes V_{\mathbf{L}}^{v\rho - \rho}$ , so by Proposition 2.17(ii) we have  $\text{Ext}^2(V_{\mathbf{L}}^{\lambda}, V_{\mathbf{L}}^{\mu}) \neq 0$ . On the other hand,  $\xi = (1, 1, 1, 0)$ , so  $(\xi, \lambda) - (\xi, \mu) = 6 - 3 = 3$ . So, in the algebra  $A_{\mathbf{B}_1}$  its bigrading is  $(2, 3)$ , while the first (in the cohomological grading) component of the algebra has bigrading  $(1, 1)$  by Lemma 3.3. Thus, the algebra cannot be one-generated, and in particular, it is not Koszul.

On the other hand, it is not difficult to check that the objects  $\mathcal{E}^{\lambda}$  with  $\lambda \in \mathbf{B}_1$  are still vector bundles. To be more precise one can check that  $\mathcal{E}^{\lambda_1, 1, 1; \lambda_4}$  is an extension of  $\mathcal{U}^{\lambda_1, 1, 1; \lambda_4}$  by an extension of  $\mathcal{U}^{\lambda_1 - 1, 1, 1; 1 - \lambda_4}$  by  $\mathcal{U}^{\lambda_1 - 2, 1, 1; \lambda_4}$ . The key point for such a verification is the fact that the non-trivial element in  $\text{Ext}^2(\mathcal{U}^{\lambda_1, 1, 1; \lambda_4}, \mathcal{U}^{\lambda_1 - 3, 1, 1; 1 - \lambda_4})$  is the Massey product of the canonical elements in  $\text{Ext}^1(\mathcal{U}^{\lambda_1, 1, 1; \lambda_4}, \mathcal{U}^{\lambda_1 - 1, 1, 1; 1 - \lambda_4})$ ,  $\text{Ext}^1(\mathcal{U}^{\lambda_1 - 1, 1, 1; 1 - \lambda_4}, \mathcal{U}^{\lambda_1 - 2, 1, 1; \lambda_4})$ , and  $\text{Ext}^1(\mathcal{U}^{\lambda_1 - 2, 1, 1; \lambda_4}, \mathcal{U}^{\lambda_1 - 3, 1, 1; 1 - \lambda_4})$ .

The above example leads to the following Conjecture.

**Conjecture 9.6.** *The algebra  $A_{\mathbf{B}}$  is one-generated as an  $A_{\infty}$  algebra. Its Koszul dual is a usual algebra.*

**9.4. Numbers of objects.** It turns out that the above collections contain the maximal possible number of objects. Recall that the rank of the Grothendieck group of  $\mathbf{G}/\mathbf{P}$  is equal to  $|\mathbf{W}_{\mathbf{G}}/\mathbf{W}_{\mathbf{L}}|$ . In the case of the series  $B$ ,  $C$  and  $D$  these ranks are given respectively by

$$\begin{aligned} r^B(n, k) &= \begin{cases} \binom{n}{k} \cdot 2^k, & k < n, \\ 2^n, & k = n, \end{cases} \\ r^C(n, k) &= \binom{n}{k} \cdot 2^k, \\ r^D(n, k) &= \binom{n}{k} \cdot 2^k, \text{ for } k \leq n - 2 \end{aligned}$$

(as was explained above, we do not need to consider the case of type  $D$  and  $k = n - 1$  or  $n$ ).

**Proposition 9.7.** *The total number of objects in the collections of Theorems 9.1, 9.2 and 9.3 equals the rank of the Grothendieck group of the corresponding Grassmannian.*

*Proof.* Let us denote

$$c_k(n) = |\{n \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0, \lambda_i \in \mathbb{Z}\}| = \binom{n+k}{k}.$$

We will consider the types  $B$ ,  $C$  and  $D$  separately.

**1. Type  $B_n$ ,  $k \leq n - 1$ .** In this case we have

$$\begin{aligned} |\bar{\mathbf{B}}_t| &= c_t(2n - k - 1 - t) c_{n-k}(\lfloor (k - t)/2 \rfloor), & \text{for integer } 0 \leq t \leq k - 1 \\ |\bar{\mathbf{B}}_{t+1/2}| &= c_t(2n - k - 1 - t) c_{n-k}(\lfloor (k - t - 1)/2 \rfloor), & \text{for integer } 0 \leq t \leq k - 1 \\ |\bar{\mathbf{B}}_t| &= c_{k-1}(2n - k - 1 - t), & \text{for integer } k \leq t \leq 2n - k - 1 \end{aligned}$$

Hence, the total number of objects in the collection of Theorem 9.1 in this case is

$$N^B(n, k) = \sum_{t=0}^{k-1} c_t(2n - k - 1 - t) \cdot (c_{n-k}(\lfloor (k - t)/2 \rfloor) + c_{n-k}(\lfloor (k - t - 1)/2 \rfloor)) + \sum_{t=k}^{2n-k-1} c_{k-1}(2n - k - 1 - t).$$

But

$$\sum_{t=k}^{2n-k-1} c_{k-1}(2n-k-1-t) = \sum_{i=0}^{2n-2k-1} c_{k-1}(i) = \sum_{i=0}^{2n-2k-1} \binom{k-1+i}{k-1} = \binom{2n-k-1}{k} = c_k(2n-2k-1).$$

Thus,

$$\begin{aligned} N^B(n, k) &= \sum_{t=0}^{k-1} c_t(2n-k-1-t) \cdot (c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_{n-k}(\lfloor (k-t-1)/2 \rfloor)) + c_k(2n-2k-1) = \\ &= \sum_{t=0}^k \binom{2n-k-1}{t} \cdot (c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_{n-k}(\lfloor (k-t-1)/2 \rfloor)). \end{aligned}$$

Hence,  $N^B(n, k)$  is the coefficient of  $x^k$  in  $(1+x)^{2n-k-1} f_{n-k}^B(x)$ , where

$$f_{n-k}^B(x) = \sum_{i \geq 0} (c_{n-k}(\lfloor i/2 \rfloor) + c_{n-k}(\lfloor (i-1)/2 \rfloor)) x^i = (1+2x+x^2) \cdot \sum_{j \geq 0} c_{n-k}(j) x^{2j} = \frac{(1+x)^2}{(1-x^2)^{n-k+1}}.$$

Therefore,  $N^B(n, k)$  is the coefficient of  $x^k$  in

$$\frac{(1+x)^{2n-k+1}}{(1-x^2)^{n-k+1}} = \frac{(1+x)^n}{(1-x)^{n-k+1}} = (1+x)^n \cdot \sum_{i \geq 0} \binom{n-k+i}{i} x^i.$$

Finally, this gives

$$N^B(n, k) = \sum_{i=0}^k \binom{n}{k-i} \binom{n-k+i}{i} = \sum_{i=0}^k \frac{n!}{(k-i)! i! (n-k)!} = \binom{n}{k} \cdot \sum_{i=0}^k \binom{k}{i} = \binom{n}{k} \cdot 2^k.$$

**1'. Type  $B_n$ ,  $k = n$ .** In this case

$$|\bar{B}_{2t}| = |\bar{B}_{2t+1}| = c_t(n-t-1) = \binom{n-1}{t},$$

and the total number of objects is

$$N^B(n, n) = 2 \sum_{t=0}^{n-1} \binom{n-1}{t} = 2 \cdot 2^{n-1} = 2^n.$$

**2. Type  $C_n$ .** We have

$$\begin{aligned} |\bar{B}_t| &= c_t(2n-k-t) \cdot c_{n-k}(\lfloor (k-t)/2 \rfloor), & \text{for integer } 0 \leq t \leq k-1 \\ |\bar{B}_t| &= c_{k-1}(2n-k-t), & \text{for integer } k \leq t \leq 2n-k \end{aligned}$$

Thus, the total number of objects is

$$\begin{aligned} N^C(n, k) &= \sum_{t=0}^{k-1} c_t(2n-k-t) \cdot c_{n-k}(\lfloor (k-t)/2 \rfloor) + \sum_{t=k}^{2n-k} c_{k-1}(2n-k-t) = \\ &= \sum_{t=0}^{k-1} c_t(2n-k-t) \cdot c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_k(2n-2k) = \sum_{t=0}^k c_t(2n-k-t) \cdot c_{n-k}(\lfloor (k-t)/2 \rfloor). \end{aligned}$$

In other words,  $N^C(n, k)$  is the coefficient of  $x^k$  in  $(1+x)^{2n-k} f_{n-k}^C(x)$ , where

$$f_{n-k}^C(x) = \sum_{i \geq 0} c_{n-k}(\lfloor i/2 \rfloor) x^i = (1+x) \sum_{j \geq 0} c_{n-k}(j) x^{2j} = \frac{(1+x)}{(1-x^2)^{n-k+1}}.$$

Therefore,  $N^C(n, k)$  is the coefficient of  $x^k$  in  $(1+x)^{2n-k+1} \cdot (1-x^2)^{-(n-k+1)}$ , so we get

$$N^C(n, k) = N^B(n, k) = \binom{n}{k} \cdot 2^k.$$

**3. Type  $D_n$ ,  $k \leq n-2$ .** First, we observe that

$$s_k(n) := |\{n \geq \lambda_1 \geq \dots \geq \lambda_k \geq -\lambda_{k-1}, \lambda_i \in \mathbb{Z}\}| = \sum_{p \geq 0} |\{n \geq \lambda_1 \geq \dots \geq \lambda_{k-1} = p, \lambda_i \in \mathbb{Z}\}| \cdot (2p+1) = \sum_{p \geq 0} (2p+1)c_{k-2}(n-p),$$

and so

$$\sum_{n \geq 0} s_k(n)x^n = \left( \sum_{p \geq 0} (2p+1)x^p \right) \cdot \frac{1}{(1-x)^{k-1}} = \frac{1+x}{(1-x)^{k+1}}.$$

Similarly,

$$t_k(n) := |\{n+1/2 \geq \lambda_1 \geq \dots \geq \lambda_k \geq -\lambda_{k-1}, \lambda_i \in 1/2 + \mathbb{Z}\}| = \sum_{p \geq 0} |\{n+1/2 \geq \lambda_1 \geq \dots \geq \lambda_{k-1} = p+1/2, \lambda_i \in 1/2 + \mathbb{Z}\}| \cdot (2p+2) = \sum_{p \geq 0} (2p+2)c_{k-2}(n-p),$$

and so

$$\sum_{n \geq 0} t_k(n)x^n = \left( \sum_{p \geq 0} (2p+2)x^p \right) \cdot \frac{1}{(1-x)^{k-1}} = \frac{2}{(1-x)^{k+1}}.$$

Now

$$\begin{aligned} |\bar{\mathbf{B}}_t| &= c_t(2n-k-2-t)s_{n-k}(\lfloor (k-t)/2 \rfloor), & \text{for integer } 0 \leq t \leq k-1 \\ |\bar{\mathbf{B}}_{t+1/2}| &= c_t(2n-k-2-t)t_{n-k}(\lfloor (k-t-1)/2 \rfloor), & \text{for integer } 0 \leq t \leq k-1 \\ |\bar{\mathbf{B}}_t| &= c_{k-1}(2n-k-2-t), & \text{for integer } k \leq t \leq 2n-k-2 \end{aligned}$$

Hence, the total number is

$$\begin{aligned} N^D(n, k) &= \sum_{t=0}^{k-1} c_t(2n-k-2-t) \cdot (s_{n-k}(\lfloor (k-t)/2 \rfloor) + t_{n-k}(\lfloor (k-t-1)/2 \rfloor)) + \sum_{t=k}^{2n-k-2} c_{k-1}(2n-k-2-t) = \\ &= \sum_{t=0}^k c_t(2n-k-2-t) \cdot (s_{n-k}(\lfloor (k-t)/2 \rfloor) + t_{n-k}(\lfloor (k-t-1)/2 \rfloor)). \end{aligned}$$

Thus,  $N^D(n, k)$  is the coefficient of  $x^k$  in  $(1+x)^{2n-k-2} f_{n-k}^D(x)$ , where

$$\begin{aligned} f_{n-k}^D(x) &= \sum_{i \geq 0} (s_{n-k}(\lfloor i/2 \rfloor) + t_{n-k}(\lfloor (i-1)/2 \rfloor)) x^i = \\ &= (1+x) \cdot \sum_{j \geq 0} s_{n-k}(j)x^{2j} + x(1+x) \cdot \sum_{j \geq 0} t_{n-k}(j)x^{2j} = \\ &= \frac{(1+x)(1+x^2)}{(1-x^2)^{n-k+1}} + \frac{2(1+x)x}{(1-x^2)^{n-k+1}} = \frac{(1+x)^3}{(1-x^2)^{n-k+1}}. \end{aligned}$$

Therefore,  $N^D(n, k)$  is the coefficient of  $x^k$  in  $(1+x)^{2n-k+1}(1-x^2)^{-(n-k+1)}$  which gives

$$N^D(n, k) = N^B(n, k) = \binom{n}{k} \cdot 2^k.$$

This completes the proof. □

**9.5. Proofs.** Here we explain how the results of the paper imply the Theorems from the Introduction.

*Proof of Theorem 1.2.* The exceptional collections are constructed in Theorems 9.1, 9.2, and 9.3. They have equivariant structure by construction. The number of objects equals the rank of the Grothendieck group by Proposition 9.7.  $\square$

*Proof of Corollary 1.3.* Recall that  $Y = \mathcal{G} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) = (\mathcal{G} \times (\mathbf{G}/\mathbf{P}))/\mathbf{G}$ , with respect to the natural right action of  $\mathbf{G}$  on  $\mathcal{G}$  and the left action on  $\mathbf{G}/\mathbf{P}$ . By [El], Theorem 9.6, the derived category  $\mathcal{D}(Y)$  is equivalent to  $\mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))^{\mathbf{G}}$ , the category of  $\mathbf{G}$ -equivariant objects in  $\mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))$ . Consider the object  $\mathcal{O}_{\mathcal{G}} \boxtimes \mathcal{E}^{\lambda} \in \mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))$  with its natural  $\mathbf{G}$ -equivariant structure. By the above observation it gives an object  $\mathcal{E}_Y^{\lambda} \in \mathcal{D}(Y)$  such that for any point  $x \in X$  we have

$$(\mathcal{E}_Y^{\lambda})|_{p^{-1}(x)} \cong \mathcal{E}^{\lambda}.$$

Thus we can apply Theorem 3.1 from [S07] and conclude that the functors

$$\Phi^{\lambda} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y), \quad F \mapsto p^*F \otimes \mathcal{E}_Y^{\lambda}$$

are fully faithful and subcategories  $\Phi_{\lambda}(\mathcal{D}(X)) \subset \mathcal{D}(Y)$  are semiorthogonal. This means that we have a semiorthogonal decomposition

$$\mathcal{D}^b(Y) = \langle \{\Phi^{\lambda}(\mathcal{D}(X))\}_{\lambda \in \mathbf{B}}, \mathcal{A} \rangle,$$

where  $\mathcal{A} = \cap_{\lambda \in \mathbf{B}} {}^{\perp}\Phi^{\lambda}(\mathcal{D}(X))$ . Now if  $X$  has an exceptional collection  $F_i$  of length  $N = \text{rk } K_0(X)$  then the objects  $p^*F_i \otimes \mathcal{E}_Y^{\lambda}$  form an exceptional collection of length  $N \cdot \#\mathbf{B}$  in  $\mathcal{D}(Y)$ , so if  $\#\mathbf{B} = \text{rk } K_0(\mathbf{G}/\mathbf{P})$  then this number equals  $\text{rk } K_0(X) \cdot \text{rk } K_0(\mathbf{G}/\mathbf{P}) = \text{rk } K_0(Y)$ , so we have an exceptional collection of expected length on  $Y$ .  $\square$

*Proof of Theorem 1.5.* Part (i) is given by Theorem 5.10. Part (ii) follows from Proposition 3.13 combined with Proposition 6.3 and Theorem 7.1. Part (iii) is a combination of Theorems 9.1, 9.2, and 9.3 with Proposition 9.7.  $\square$

*Proof of Theorem 1.8.* This is just Proposition 4.2.  $\square$

**9.6. Usual Grassmannians.** In this section we speculate that our construction might still work with a certain weakening of the assumption (24) (so that  $D_{\text{out}}$  is not necessarily connected). Namely, we consider the case  $X = \text{Gr}(k, n)$ , the usual Grassmannian, and apply formally the procedure of section 5 to the data for which (24) does not hold to construct collections of expected length in  $\mathcal{D}^b(X)$ . Of course, our proof of part (b) of the criterion of exceptionality (see Proposition 3.13) does not work in this situation, so we do not have a proof of exceptionality of this collection. However, we believe that all these collections are exceptional and full.

Since the result of this section is only conjectural, we skip the intermediate calculations (which are analogous to those for isotropic Grassmannians) and only state the final answer.

Let  $\mathbf{G} = \text{SL}_n$  and  $\mathbf{L} = (\text{GL}_k \times \text{GL}_l) \cap \text{SL}_n$  ( $n = k + l$ ). In the framework of the paper we could take  $D_{\text{out}}$  to be either of the two connected components of  $D_{\mathbf{G}} \setminus \beta$ . Let us take instead  $D_{\text{out}}$  to be the union of both, that is  $D_{\text{out}} = D_{\mathbf{G}} \setminus \beta$ . Of course we violate here the assumption (24).

Moreover, we arbitrarily renumber the vertices of  $D_{\mathbf{G}}$  in such a way that  $D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}$  is always connected and contains  $\beta = \alpha_{n-1}$ . In other words, to obtain from  $D_{\mathbf{G}}$  the chain of Dynkin diagrams  $D_a$  we keep chopping off one of the end-points of the diagram until only  $\beta$  is left.

It is clear that such renumberings are in a bijection with isotopy classes of monotone curves  $C$  in a  $k \times l$  rectangle on an integer grid going from the point  $(k, l)$  to the point  $(0, 0)$  and not passing through integer points. We will describe a conjectural exceptional collection corresponding to an isotopy class of such a curve.

Moreover, in fact we will allow the curve to pass through integer points (this corresponds to allowing to chop off both end-points simultaneously).

So, assume we are given such a curve  $C$ . Consider the sequence of points  $Q_0, Q_1, \dots, Q_m$  of intersection of  $C$  with the edges of the grid squares (some of the points  $Q_i$  can lie at the vertices of the squares) and let  $(x_i, y_i)$  be the coordinates of  $Q_i$ . Set

$$a_i = \lfloor x_i \rfloor, \quad b_i = \lfloor y_i \rfloor, \quad c_i = k - \lceil x_i \rceil, \quad d_i = l - \lceil y_i \rceil.$$

Then consider the blocks

$$(57) \quad B_i = \left\{ \begin{array}{l} d_i + i \geq \lambda_1 \geq \dots \geq \lambda_{a_i} \geq i = \lambda_{a_i+1} = \dots = \lambda_k, \\ \lambda_{k+1} = \dots = \lambda_{n-b_i} = 0 \geq \lambda_{n-b_i+1} \geq \dots \geq \lambda_n \geq -c_i \end{array} \right\}.$$

(in particular,  $B_0 = \{0\}$ ). Note that the total number of weights in those blocks is

$$\#(B_0 \sqcup B_1 \sqcup \dots \sqcup B_m) = \sum_{i=0}^m \binom{a_i + d_i}{a_i} \binom{b_i + c_i}{b_i} = \binom{k+l}{k},$$

which is the rank of the Grothendieck group of  $X = \text{Gr}(k, n)$ . The equality above has a simple combinatorial proof — the RHS is the number of Young diagrams inscribed in the rectangle, we divide the set of all such diagrams into subsets numbered by the point of intersection of the border of the diagram with the curve  $C$ , the summands in the LHS correspond to the parts of this decomposition.

We have the following

**Conjecture 9.8.** *The collection  $\langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$  with subcategories  $\mathcal{A}_i = \langle \mathcal{U}^\lambda \rangle_{\lambda \in B_i}$  and blocks  $B_i$  given by (57) is a semiorthogonal decomposition of  $\mathcal{D}^b(\text{Gr}(k, n))$ , each component of which is generated by an exceptional collection.*

*Remark 9.9.* One special case is interesting. Assume  $l = k$ , and take for  $C$  the segment of the straight line from  $(k, k)$  to  $(0, 0)$ . Then  $m = k$  and  $Q_i = (i, i)$  so that  $a_i = b_i = i$ ,  $c_i = d_i = k - i$ . The corresponding exceptional collection is invariant with respect to the outer automorphism of  $\text{Gr}(k, 2k)$ .

## 10. APPENDIX. THE KEY-PROPOSITION

For a dominant weight  $\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$  of  $\text{GL}_n$  we write  $\lambda \geq 0$  (and say that  $\lambda$  is **nonnegative**) if  $\lambda_n \geq 0$ . Such weights correspond to partitions. Let  $w_0$  denote the longest element of the symmetric group  $S_n$ . The goal of this Appendix is to prove the following

Let  $\sigma$  be a partition with at most  $n - a$  parts. Denote by

$$\Pi_\sigma : \text{Rep} - \text{GL}_n \rightarrow \text{Rep} - \text{GL}_n$$

the projector which acts identically on all representations with highest weight  $\nu - w_0\sigma$  with  $\nu$  being a partition with at most  $a$  parts, and by zero on all other irreducible representations. We say that a morphism  $f : V_1 \rightarrow V_2$  of representations of  $\text{GL}_n$  is a  $\sigma$ -isomorphism if  $\Pi_\sigma(f) : \Pi_\sigma(V_1) \rightarrow \Pi_\sigma(V_2)$  is an isomorphism.

**Proposition 10.1.** *Fix an integer  $a$ ,  $0 \leq a \leq n$ . Let  $\kappa$  be a partition with at most  $a$  parts, and let  $\sigma, \tau$  be partitions with at most  $n - a$  parts (all viewed as weights of  $\text{GL}_n$ ). Finally, let  $W$  be a representation which is a direct summand of  $V^{\otimes N}$ . In other words, we assume that the highest weights of all irreducible summands of  $W$  are nonnegative. Then the natural maps*

$$(58) \quad V^{\kappa - w_0\tau} \otimes W \rightarrow V^\kappa \otimes V^{-w_0\tau} \otimes W \rightarrow V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \quad \text{and}$$

$$(59) \quad V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \rightarrow V^\kappa \otimes V^{-w_0\tau} \otimes W \rightarrow V^{\kappa - w_0\tau} \otimes W$$

are  $\sigma$ -isomorphisms.

*Remark 10.2.* For the purposes of this paper we need only the first part of the above Proposition. However, since the situation is pretty symmetrical and the proof is the same we prove both parts.

We will only use the following corollary of the case  $\sigma = 0$  of the above Proposition.

**Corollary 10.3.** *Fix  $a$ ,  $0 \leq a \leq n-1$ . Let  $\kappa$  and  $\nu$  be partitions with  $\leq a$  parts,  $\tau$  a partition with  $\leq n-a$  parts, and  $\mu$  a partition with  $\leq n$  parts. For  $l \geq 0$  let  $\Pi_{-l}^a$  be the projector on the category of  $\mathrm{GL}_n$ -representations which acts identically on  $V^\lambda$ , where  $\lambda_{a+1} = \dots = \lambda_n = -l$ , and sends all other irreducible representations to zero. Then the natural map*

$$V^{\kappa-w_0\tau} \otimes V^\mu \rightarrow V^\kappa \otimes V^{-w_0\tau} \otimes V^\mu \rightarrow V^\kappa \otimes \Pi_{-l}^a(V^{-w_0\tau} \otimes V^\mu)$$

*induces an isomorphism after applying  $\Pi_{-l}^a$ .*

*Proof.* Replacing all  $\mu_i$  by  $\mu_i + l$  we can assume that  $l = 0$ . But in this case the assertion follows from Proposition 10.1 with  $\sigma = 0$ .  $\square$

The proof of Proposition 10.1 will take the rest of the section. We start with the following simple reduction:

**Lemma 10.4.** *Let  $\kappa, \sigma, \tau$  and  $W$  be as in the Proposition. Assume that the map*

$$(60) \quad \begin{array}{ccc} \Pi_\sigma(V^{\kappa-w_0\tau} \otimes W) & & V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \\ \downarrow & & \uparrow \\ V^{\kappa-w_0\tau} \otimes W & \xrightarrow{\quad} & V^\kappa \otimes V^{-w_0\tau} \otimes W \end{array}$$

*is injective. Then the map (58) is a  $\sigma$ -isomorphism. Similarly, if the map*

$$(61) \quad \begin{array}{ccc} V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) & & \Pi_\sigma(V^{\kappa-w_0\tau} \otimes W) \\ \downarrow & & \uparrow \\ V^\kappa \otimes V^{-w_0\tau} \otimes W & \xrightarrow{\quad} & V^{\kappa-w_0\tau} \otimes W \end{array}$$

*is surjective then the map (59) is a  $\sigma$ -isomorphism.*

*Proof.* It is enough to check that the multiplicities of irreducible summands with highest weights of the form  $\mu - w_0\sigma$  with  $\mu$  being a partition with no more than  $a$  parts, in the source and the target of the composed map (60) (resp., (61)) are equal. To this end we replace  $W$  with any of its irreducible summand of the form  $V^\lambda$ , where  $\lambda \geq 0$ , and apply the Littlewood-Richardson rule. The dimension of  $\mathrm{Hom}(V^{\mu-w_0\sigma}, V^{\kappa-w_0\tau} \otimes V^\lambda)$  is given by the number of semistandard skew tableaux  $S$  of shape  $(\mu - w_0\sigma) \setminus (\kappa - w_0\tau)$  with the content of weight  $\lambda$ , satisfying the lattice permutation condition. Every such skew tableau contains a skew subtableau  $S'$  of shape  $\mu \setminus \kappa$  that still satisfies the lattice permutation condition. Let  $\nu$  be the weight of the content of  $S'$ . Then to give  $S$  is the same as to give  $\nu \subset \lambda$  together with a pair:

- (i) a semistandard skew tableau of shape  $\mu \setminus \kappa$  with content of weight  $\nu$ ,
- (ii) a semistandard skew tableau of shape  $(\nu - w_0\sigma) \setminus (-w_0\tau)$  with content  $\lambda$ .

Let  $N_1$  (resp.,  $N_2$ ) be the number of choices in (i) (resp., in (ii)). We have

$$N_1 = \dim \mathrm{Hom}(V^\mu, V^\kappa \otimes V^\nu).$$

Applying the Littlewood-Richardson rule with  $\kappa$  and  $\nu$  switched we can interpret this as the number of semistandard skew tableau of shape  $\mu \setminus \nu$  with content of weight  $\kappa$ . Since  $\mu \setminus \nu = (\mu - w_0\sigma) \setminus (\nu - w_0\sigma)$ , we deduce that

$$N_1 = \dim \mathrm{Hom}(V^{\mu-w_0\sigma}, V^\kappa \otimes V^{\nu-w_0\sigma}).$$

On the other hand,

$$N_2 = \dim \operatorname{Hom}(V^{\nu-w_0\sigma}, V^{-w_0\tau} \otimes V^\lambda).$$

Thus, if we vary  $\nu$  we get dimension of

$$\bigoplus_{\nu \geq 0, \nu \subset \mu, \nu \subset \lambda} \operatorname{Hom}(V^{\mu-w_0\sigma}, V^\kappa \otimes V^{\nu-w_0\sigma}) \otimes \operatorname{Hom}(V^{\nu-w_0\sigma}, V^{-w_0\tau} \otimes V^\lambda) = \operatorname{Hom}(V^{\mu-w_0\sigma}, V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes V^\lambda)).$$

Here the last equality follows from the fact that for a partition  $\nu \geq 0$  with no more than  $a$  parts and nonzero  $\operatorname{Hom}(V^{\mu-w_0\sigma}, V^\kappa \otimes V^{\nu-w_0\sigma})$  (resp.,  $\operatorname{Hom}(V^{\nu-w_0\sigma}, V^{-w_0\tau} \otimes V^\lambda)$ ) one necessarily has  $\nu \subset \mu$  (resp.,  $\nu \subset \lambda$ ).  $\square$

So, to deduce the Proposition it suffices to check injectivity (resp. surjectivity) of the map in question. Before considering the general case we start with two simple cases.

**Lemma 10.5.** *If  $\sigma = \tau$  then the composition (60) is injective (resp., the composition (61) is surjective).*

*Proof.* Indeed, the composition of the first two arrows in (60) is an embedding. On the other hand, the third arrow is a projection onto a direct summand, and the other summands do not have irreducible components which can be written as  $\nu - w_0\tau$ . The case of (61) is similar.  $\square$

Another simple case is the following.

**Lemma 10.6.** *Assume that the Young diagram  $\sigma$  is obtained from  $\tau$  by removing one box and  $W = V$ . Then the composition (60) is injective (resp., the composition (61) is surjective).*

*Proof.* Indeed, in this case (60) looks like

$$(62) \quad \begin{array}{ccc} V^{\kappa-w_0\sigma} & & V^\kappa \otimes V^{-w_0\sigma} \\ \downarrow & & \uparrow \\ V^{\kappa-w_0\tau} \otimes V & \longrightarrow & V^\kappa \otimes V^{-w_0\tau} \otimes V \end{array}$$

Let us check that the composition sends the highest weight vector  $v_{\kappa-w_0\sigma}$  to a nonzero multiple of the highest weight vector  $v_\kappa \otimes v_{-w_0\sigma}$ .

First, let us introduce some notation. Let  $u_1, \dots, u_n$  be the standard weight basis in  $V$ . Let also  $e_k, f_k, h_k$  be the standard Chevalley generators of the Lie algebra  $\mathfrak{sl}_n$ , so that

$$e_k u_i = \begin{cases} u_k, & \text{if } i = k+1 \\ 0, & \text{otherwise} \end{cases} \quad f_k u_i = \begin{cases} u_{k+1}, & \text{if } i = k \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h_k = [e_k, f_k].$$

Denote also

$$f_{[j,i-1]} := f_j \cdot f_{j+1} \cdots f_{i-1} \in U(\mathfrak{sl}_n).$$

Then one can easily see that we have

$$(63) \quad [e_k, f_{[j,i-1]}] = \begin{cases} f_{[j,i-2]}, & \text{for } k = i-1 \neq j \\ -f_{[j+1,i-1]}, & \text{for } k = j \neq i-1 \\ h_k, & \text{for } k = j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

Let  $\pi_\lambda : V^\lambda \rightarrow \mathbb{C}$  denote the projection onto the highest weight component. It is well known that a singular vector in  $V^\lambda \otimes V^\mu$  is uniquely determined by its image under  $\pi_\lambda \otimes \operatorname{id} : V^\lambda \otimes V^\mu \rightarrow V^\mu$ . We claim that the unique singular vector  $\tilde{v}_{\kappa-w_0\sigma} \in V^{\kappa-w_0\tau} \otimes V$  of weight  $\kappa - w_0\sigma$  can be rescaled so that

$$(\pi_{\kappa-w_0\sigma} \otimes \operatorname{id})(\tilde{v}_{\kappa-w_0\sigma}) = u_i,$$



where  $i$  is the number of the component in which  $-w_0\tau$  and  $-w_0\sigma$  are distinct (that is  $\sigma$  is obtained from  $\tau$  by removing a box from  $(n+1-i)$ -th column). More precisely, we have

$$(64) \quad \tilde{v}_{\kappa-w_0\sigma} = v_{\kappa-w_0\tau} \otimes u_i - f_{i-1}v_{\kappa-w_0\tau} \otimes u_{i-1} - \cdots - f_{[1,i-1]}v_{\kappa-w_0\tau} \otimes u_1.$$

Indeed, it is enough to check that the right-hand side is annihilated by all  $e_k$  which can be verified by a direct computation using relations (63).

Note that in the case  $\kappa = 0$  we obtain that the unique singular vector  $\tilde{v}_{-w_0\sigma}$  of weight  $-w_0\sigma$  in  $V^{-w_0\tau} \otimes V$  is given by

$$\tilde{v}_{-w_0\sigma} = v_{-w_0\tau} \otimes u_i - f_{i-1}v_{-w_0\tau} \otimes u_{i-1} - \cdots - f_{[1,i-1]}v_{-w_0\tau} \otimes u_1.$$

Now the first arrow in the composition (62) sends  $v_{\kappa-w_0\sigma}$  to the singular vector  $\tilde{v}_{\kappa-w_0\sigma}$  given by (64). The second arrow sends  $\tilde{v}_{\kappa-w_0\sigma}$  to

$$x = v_{\kappa} \otimes v_{-w_0\tau} \otimes u_i - f_{i-1}(v_{\kappa} \otimes v_{-w_0\tau}) \otimes u_{i-1} - \cdots - f_{[1,i-1]}(v_{\kappa} \otimes v_{-w_0\tau}) \otimes u_1 \in V^{\kappa} \otimes V^{-w_0\tau} \otimes V.$$

Note that

$$(\pi_{\kappa} \otimes \text{id} \otimes \text{id})(x) = \tilde{v}_{-w_0\sigma} \in V^{-w_0\tau} \otimes V.$$

The projection  $V^{-w_0\tau} \otimes V \rightarrow V^{-w_0\sigma}$  sends  $\tilde{v}_{-w_0\sigma}$  to the highest weight vector  $v_{-w_0\sigma} \in V^{-w_0\sigma}$ . Therefore, the third arrow in the composition (62) sends  $x$  to a highest weight vector  $y \in V^{\kappa} \otimes V^{-w_0\sigma}$ , such that  $\pi_{\kappa}(y) = v_{-w_0\sigma}$ . Hence,  $y = v_{\kappa} \otimes v_{-w_0\sigma}$  as required.

Now let us prove surjectivity of the composition (61) in our case. We will check that the composition

$$\begin{array}{ccc} V^{\kappa} \otimes V^{-w_0\sigma} & & V^{\kappa-w_0\sigma} \\ \downarrow & & \uparrow \\ V^{\kappa} \otimes V^{-w_0\tau} \otimes V & \longrightarrow & V^{\kappa-w_0\tau} \otimes V \end{array}$$

sends the highest weight vector  $v_{\kappa} \otimes v_{-w_0\sigma}$  to a nonzero multiple of the highest weight vector  $v_{\kappa-w_0\sigma}$ . The first map sends  $v_{\kappa} \otimes v_{-w_0\sigma}$  to a nonzero multiple of  $v_{\kappa} \otimes \tilde{v}_{-w_0\sigma}$ , where  $(\pi_{-w_0\tau} \otimes \text{id})(\tilde{v}_{-w_0\sigma}) = u_i$ . Let  $\rho : V^{\kappa} \otimes V^{-w_0\tau} \rightarrow V^{\kappa-w_0\tau}$  denote the natural map. Since

$$\pi_{\kappa-w_0\tau} = \pi_{\kappa} \otimes \pi_{-w_0\tau},$$

it follows that

$$(\pi_{\kappa-w_0\tau} \otimes \text{id})(\rho \otimes \text{id})(v_{\kappa} \otimes \tilde{v}_{-w_0\sigma}) = u_i.$$

Hence,

$$(\rho \otimes \text{id})(v_{\kappa} \otimes \tilde{v}_{-w_0\sigma}) = \tilde{v}_{\kappa-w_0\sigma},$$

which projects to a nonzero multiple of  $v_{\kappa-w_0\sigma}$  in  $V^{\kappa-w_0\sigma}$ .  $\square$

Now we can prove the Proposition.

*Proof.* We will only consider the case of the map (58). The case of the map (59) is analogous using the corresponding parts of Lemmas 10.5 and 10.6.

First of all, consider the case  $W = V$ . Then the composition (60) is either the map of Lemma 10.5, or the map of Lemma 10.6. In both cases we know that it is injective.

Now let us check that the injectivity of the map (60) for  $W$  implies its injectivity for  $W' = V \otimes W$ . Indeed, let

$$(65) \quad V^{\kappa} \otimes V = \bigoplus V^{\kappa_i} \oplus (V^{\kappa'}),$$

where all  $\kappa_i$  are partitions with at most  $a$ -parts, and  $\kappa'$  is the partition with  $(a+1)$  parts obtained from  $\kappa$  by adding a summand equal to 1 (this last summand may be missing). Let also

$$(66) \quad V^{-w_0\tau} \otimes V = \bigoplus V^{-w_0\tau_i} \oplus V^{\omega_1-w_0\tau}.$$

Then it is clear that

$$(67) \quad V^{\kappa-w_0\tau} \otimes V = \bigoplus V^{\kappa_i-w_0\tau} \oplus \bigoplus V^{\kappa-w_0\tau_i} \oplus (V^{\kappa'-w_0\tau})$$

(the last summand is present only if  $\tau$  has less than  $n-a$  parts and  $\kappa$  has precisely  $a$  parts). Note that if the last summand is present then for any  $W$  we have  $\Pi_\sigma(V^{\kappa'-w_0\tau} \otimes W) = 0$ , since  $V^\lambda \subset V^{\kappa'-w_0\tau} \otimes W$  we have  $\lambda_{a+1} > 0$ . Hence, we can omit this last summand when we apply  $\Pi_\sigma$ .

Consider the following commutative diagram

$$\begin{array}{ccc}
\Pi_\sigma(\bigoplus V^{\kappa_i-w_0\tau} \otimes W) \oplus \Pi_\sigma(\bigoplus V^{\kappa-w_0\tau_i} \otimes W) & \xrightarrow{(67)} & \Pi_\sigma(V^{\kappa-w_0\tau} \otimes V \otimes W) \\
\downarrow & & \downarrow \\
\bigoplus V^{\kappa_i} \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \oplus V^\kappa \otimes \Pi_\sigma(\bigoplus V^{-w_0\tau_i} \otimes W) & & V^\kappa \otimes \Pi_\sigma(V^{-w_0\tau} \otimes V \otimes W) \\
\downarrow (65) & & \parallel (66) \\
V^\kappa \otimes V \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \oplus V^\kappa \otimes \Pi_\sigma(\bigoplus V^{-w_0\tau_i} \otimes W) & & V^\kappa \otimes [\Pi_\sigma(V^{\omega_1-w_0\tau} \otimes W) \oplus \Pi_\sigma(\bigoplus V^{-w_0\tau_i} \otimes W)] \\
\parallel & \swarrow & \\
V^\kappa \otimes [V \otimes \Pi_\sigma(V^{-w_0\tau} \otimes W) \oplus \Pi_\sigma(\bigoplus V^{-w_0\tau_i} \otimes W)] & & 
\end{array}$$

The first arrow in the first column is injective by the induction hypothesis, and the middle vertical arrow in this column is injective by (65). Note also that the diagonal arrow is injective by the induction hypothesis, although we actually do not need to know this. Anyway, the composition of all the maps from the top right corner to the bottom left corner is injective, hence the first arrow in the second column has to be injective as well. But this is the claim for  $V \otimes W$ .

Thus, we have checked that composition (60) is injective for all  $W = V^{\otimes N}$ . It remains to check that it is injective for all direct summands of such  $W$ . Indeed, assume that  $W = V^{\otimes N} = W_1 \oplus W_2$ . Then it is clear that composition (60) for  $W$  is the direct sum of compositions (60) for  $W_1$  and  $W_2$ . Since we know this direct sum to be injective, it follows that each summand is injective as well.  $\square$

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**A.K.:** ALGEBRA SECTION, STEKLOV MATHEMATICAL INSTITUTE, 8 GUBKIN STR., MOSCOW 119991 RUSSIA

THE PONCELET LABORATORY, INDEPENDENT UNIVERSITY OF MOSCOW

LABORATORY OF ALGEBRAIC GEOMETRY, SU-HSE, 7 VAVILOVA STR., MOSCOW, RUSSIA, 117312

*E-mail address:* akuznet@mi.ras.ru

**A.P.:** DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97405

*E-mail address:* apolish@uoregon.edu